TWIN CONVERGENCE REGIONS FOR CONTINUED FRACTIONS

V. F. COWLING, WALTER LEIGHTON, AND W. J. THRON

1. Introduction. Some years ago it was proved [2][1] that the continued fraction

$$1+\frac{a_1}{1}+\frac{a_2}{1}+\cdots \tag{1.1}$$

converges if the complex numbers a_n satisfy the following conditions:

$$|a_{2n+1}| \leqq 1/4, \qquad |a_{2n}| \geqq 25/4 \qquad (n = 1, 2, \cdots). \tag{1.2}$$

Later one of the present authors proved [3] that (1.1) converges if

$$\begin{aligned} &|1+a_2| \geqq 1+|a_1|, \\ &|1+a_n+a_{n+1}| \geqq 1+|a_{n-1}a_n| \quad (n = 2, 3, \cdots). \end{aligned} \tag{1.3}$$

One of the immediate consequences of this theorem is that if

$$\begin{aligned} &|1+a_2| > 1, \qquad |a_2| \geqq (2+m)/(1-m), \\ &|a_{2n+1}| \leqq m < 1, \qquad |a_{2n+2}| \geqq 2+m+m|a_{2n}| \\ &\qquad\qquad\qquad\qquad (n = 1, 2, 3, \cdots), \end{aligned} \tag{1.4}$$

then (1.1) converges.

Recently Thron [6] has shown that if

$$|a_{2n+1}| \leqq k^2 < 1, \qquad |a_{2n}| \geqq (1+k)^2+e \qquad (e > 0), \tag{1.5}$$

the continued fraction (1.1) converges. For $k^2<1$ this result can be shown to be a "best" result except possibly for the presence of the quantity e.

The present paper is concerned with establishing convergence criteria of this general type. The principal result is given in Theorem 3.1. The method to be used is the following. Denote the nth approximant of (1.1) by A_n/B_n. Conditions on the numbers a_n are determined which imply that the approximants lie in a given region V of the complex plane. A continued fraction the elements of which are functions of the complex variable z and which reduces to (1.1) for $z=1$ is then introduced. When the given conditions on the numbers a_n are satisfied the approximants of this continued fraction are shown to

Presented to the Society, September 13, 1943, under the title *On convergence regions for continued fractions*; received by the editors June 1, 1943.

[1] Numbers in brackets refer to the Bibliography at the end of the paper.

form a normal family for z in a region D of the complex plane which includes the point $z=1$. From the known convergence of this family in a sub-region of D the (uniform) convergence of the family in every closed region interior to D may be inferred. The convergence of (1.1) is then trivial.

2. **Two lemmas.** It will be convenient to refer to an open connected set as an *open region* and to use the term *region* to mean an open region plus part or all its boundary. If S is a set of complex numbers and a is a fixed complex number, we shall designate by $a+S$ and by $S+a$ the set obtained by adding to each element of S the number a. Geometrically it is clear that $a+S$ is a translation of the set S by means of the vector a. Similarly aS will denote the set obtained from S by multiplying each element of S by the number a. Geometrically this constitutes a stretching of the set S followed by a rotation of the resulting set through an angle of arg a. By the notation $D[aS]$ is meant the point set intersection of all sets aS as a assumes all values in a given set A.

The following lemma is an extension of a result due to Scott and Wall [4].

LEMMA 2.1. *Let D_1, D_2, V_1, V_2 be any four sets of points in the complex plane which have the following properties*:

(a) $1+D_1 \subset V_1$,

(b) $1+D_2 \subset V_2$,

(c) $1+a/v \in V_1$, *if* $a \in D_1$, *and* $v \in V_2$,

(d) $1+a/v \in V_2$, *if* $a \in D_2$, *and* $v \in V_1$.

If the numbers $a_{2n-1} \in D_1$ and $a_{2n} \in D_2$ $(n=1, 2, \cdots)$, the values of the approximants of the continued fraction (1.1) *are numbers which are contained in V_1, and the values of the approximants of*

$$1+\frac{a_2}{1+}\,\frac{a_3}{1+}\cdots \tag{2.1}$$

belong to V_2.

The proof of this lemma is by an easy induction.

We proceed with the proof of the following result which is a consequence of Lemma 2.1.

LEMMA 2.2. *Let V_1 and V_2 be any two sets of numbers in the complex plane such that the sets*

$$E_1 = D[v_2(V_1-1)],$$
$$E_2 = D[v_1(V_2-1)]$$

have the property that

$$1 + E_1 \subset V_1, \qquad 1 + E_2 \subset V_2.$$

If the numbers $a_{2n-1} \in E_1$ *and* $a_{2n} \in E_2$ $(n = 1, 2, \cdots)$, *the values of the approximants of* (1.1) *belong to* V_1 *and the values of the approximants of* (2.1) *belong to* V_2.

To prove this lemma it will be sufficient to show that conditions (c) and (d) of Lemma 2.1 are satisfied when we have identified the sets D_i of that lemma with the present sets E_i, for $i = 1, 2$, respectively. We shall prove that condition (c) holds. The proof for (d) will then follow mutatis mutandis. To that end let a be any point belonging to E_1 and v_2 be any point belonging to V_2 and observe that the definition of E_1 insures the existence of a point $v'_1 \in V_1$ such that $a = v_2(v'_1 - 1)$. It follows that $1 + a/v_2 = v'_1 \in V_1$.

3. **The principal theorem.** We proceed with the proof of the following result.

THEOREM 3.1. *Let* k *be any real number greater than* 1 *and let* ϵ *and* ϵ_1 *be positive numbers less than* k. *If the elements* $a_n = r_n e^{i\theta_n}$ *are such that*

$$r_{2n} \geqq 2[k + \epsilon_1 - \cos\theta_{2n}], \qquad r_{2n-1} \leqq k - \epsilon \qquad (n = 1, 2, \cdots),$$

the continued fraction (1.1) *converges.*

The first step in the demonstration of this theorem is the proof of the following lemma.

LEMMA 3.1. *Let* k *be any real number greater than* 1. *If the elements* $a_n = r_n e^{i\theta_n}$ *are such that*

$$r_{2n-1} \leqq k, \qquad r_{2n} \geqq 2(k - \cos\theta_{2n}) \quad (0 \leqq \theta_{2n} \leqq 2\pi;\ n = 1, 2, \cdots),$$

the values of the approximants of (1.1) *belong to* V_1 *and the values of the approximants of* (2.1) *belong to* V_2.

Let the sets V_1 and V_2 of Lemma 2.2 be respectively the sets $|z-1| \leqq k$ and $|z| \geqq 1$. It will follow that (i) the set $E_1 = D[v_2(V_1 - 1)]$ is the set $|z| \leqq k$ and that (ii) $E_2 = D[v_1(V_2 - 1)]$ is the set of points $z = re^{i\theta}$ for which $r \geqq 2(k - \cos\theta)$.

To prove (i) observe that $V_1 - 1$ is the set of points z for which $|z| \leqq k$ and that the set $v'_2(V_1 - 1) \subset v_2(V_1 - 1)$ whenever v_2 and v'_2 are elements of V_2 such that $\arg v'_2 = \arg v_2$ and $|v'_2| < |v_2|$. For in these circumstances the set $v_2(V_1 - 1)$ is the circular region defined by $|z| \leqq k|v_2|$ while the set $v'_2(V_1 - 1)$ is the region $|z| \leqq k|v'_2| = k$. Thus

the set E_1 is the set $|z| \leq k$. We note that

$$1 + E_1 \subset V_1. \tag{3.1}$$

The proof of (ii) is somewhat more complicated. We observe that V_2-1 is the set of complex numbers z for which $|z+1| \geq 1$. The set $0(V-1)$ is not well-defined. From continuity considerations we shall define it to be the set exterior to the null circle $|z| = 0$. Consider then the set of products $v_1(V_2-1)$ where $v_1 \in V_1$ and $v_1 \neq 0$. If v_1 and v_1' are any two such elements of V_1 having the properties $\arg v_1 = \arg v_1'$ and $|v_1'| > |v_1|$, then $v_1'(V_2-1) \subset v_1(V_2-1)$. This is true since multiplication of V_2-1 by v_1' subjects the set to the same rotation as multiplication by v_1. Since however $|v_1'| > |v_1|$, the "exterior" set which is $v_1(V_2-1)$ contains the "exterior" set $v_1'(V_2-1)$. Thus $D[v_1'(V_2-1)]$ is the set $D[v_1(V_2-1)] = E_2$ for v_1' on the boundary of V_1. We can now prove that E_2 is the set of numbers $z = re^{i\theta}$ for which

$$r \geq 2(k - \cos\theta).$$

To determine the boundary of E_2 it is clearly sufficient to consider only those products $v_1'(v_2'-1)$ where v_1' and v_2' range over the boundaries of V_1 and V_2 respectively. To that end let $r(\alpha)e^{i\alpha}$ and $\rho(\beta)e^{i\beta}$ be arbitrary points of the boundaries of V_1 and V_2-1, respectively. It is clear that

$$r(\alpha) = \cos\alpha + (k^2 - \sin^2\alpha)^{1/2} \qquad (0 \leq \alpha \leq 2\pi),$$

$$\rho(\beta) = -2\cos\beta \qquad (\pi/2 \leq \beta \leq 3\pi/2).$$

We shall show first that

$$r(\alpha)\rho(\beta) \leq 2[k - \cos(\alpha + \beta)] \qquad (all\ \alpha, \beta). \tag{3.2}$$

It will follow then that the boundary of E_2 is not exterior to the limaçon $r = 2(k - \cos\theta)$. The proof of (3.2) is easy. It is clearly sufficient to prove that

$$-\cos\beta(k^2 - \sin^2\alpha)^{1/2} \leq k + \sin\alpha\sin\beta.$$

Since $\cos\beta < 0$ and $k > 1$, it is then sufficient to prove that

$$\cos^2\beta(k^2 - \sin^2\alpha) \leq (k + \sin\alpha\sin\beta)^2.$$

But this is equivalent to

$$0 \leq (k\sin\beta + \sin\alpha)^2. \tag{3.3}$$

Thus (3.2) is true. However, we observe that inequalities (3.2) and (3.3) are equivalent. With this in mind we can establish that corresponding to each angle $\alpha+\beta=\theta$ there exists an essentially unique

choice of α and of β such that the equality sign in (3.2) and in (3.3) holds. It will follow that the limaçon $r=2(k-\cos\theta)$ is the boundary of E_2. The angles α and β referred to are determined by the following conditions:

$$\sin\alpha = -k\sin\theta/\Delta, \qquad \cos\alpha = (1-k\cos\theta)/\Delta,$$
$$\sin\beta = \sin\theta/\Delta, \qquad \cos\beta = (\cos\theta - k)/\Delta,$$

where $\Delta=(1-2k\cos\theta+k^2)^{1/2}$.

It is clear that these equations determine α and β to within an additive integral multiple of 2π. It is easy to see that (3.3) and hence (3.2) reduces to an equality for this choice of α and β. It is not difficult to prove that there is no other choice of α and β having the desired properties. However, since the uniqueness is irrelevant to our present discussion, its proof is omitted.

The proof of the lemma is completed by noting that

$$1 + E_2 \subset V_2 \tag{3.4}$$

and recalling (3.1).

To complete the proof of Theorem 3.1 we introduce the continued fraction

$$1 + \frac{a_1 z}{1} + \frac{a_2/z}{1} + \frac{a_3 z}{1} + \frac{a_4/z}{1} + \cdots, \tag{3.5}$$

where $z=re^{i\theta}$ is a complex variable. This continued fraction reduces to (1.1) when $z=1$. We shall suppose that the coefficients a_n satisfy the hypotheses of Theorem 3.1. It is clear then that for $|z| \leqq 1+\delta_1$, where $\delta_1=\epsilon/(k-\epsilon)$,

$$|a_{2n-1}z| \leqq k. \tag{3.6}$$

Further, since the numbers a_{2n} lie outside the limaçon $r=2(k+\epsilon_1-\cos\theta)$, positive numbers η and δ_2 can be chosen so small that for $|\theta| \leqq \eta$ and $r \leqq 1+\delta_2$ we have $a_{2n}/z=Re^{i\phi}$ satisfying the conditions

$$R_{2n} \geqq 2(k - \cos\phi_{2n}).$$

Let δ be the smaller of δ_1 and δ_2. In the open region D defined by the inequalities

$$D: \quad |\theta| < \eta, \qquad 0 < |z| < 1+\delta \tag{3.7}$$

the approximants of the continued fraction (3.5) are a family of rational functions of z. By (3.6) and (3.7) Lemma 3.1 asserts that the

values of these approximants lie in the bounded region V_1. Thus this family is a normal family for z in D.

We shall now show that this normal family converges in an open subset of D. Denote by h the smaller of the two numbers $1/4(k-\epsilon)$ and $8(k+\epsilon_1-1)/25$. Then for $|z| \leqq h$, the elements of (3.5) have the property that

$$|a_{2n-1}z| \leqq 1/4, \qquad |a_{2n}/z| \geqq 25/4 \qquad (n = 1, 2, \cdots).$$

Thus by the theorem referred to concerning inequalities (1.2), the continued fraction (3.5) converges for $|z| \leqq h$ and hence in the intersection of this circular region with the region D. By Montel's [1] generalization of the Stieltjes-Vitali theorem the sequence of approximants of (3.5) converges (uniformly) in every closed region interior to D. Thus, in particular, this family converges for $z=1$.

The proof of the theorem is complete.

The following corollary is an immediate consequence of the theorem. The notation is that of the theorem.

COROLLARY 3.1. *If*

$$r_{2n} \geqq 2(k + \epsilon_1 + 1), \qquad r_{2n-1} \leqq k - \epsilon \qquad (n = 1, 2, \cdots),$$

the continued fraction (1.1) *converges.*

It is a consequence of Theorem 3.1 that if the hypotheses of this theorem hold, the continued fraction (2.1) converges at least in the wider sense, and converges to a finite value if and only if the continued fraction (1.1) converges to a value not equal to 1.

The following result is thus established. The notation is that of the theorem.

COROLLARY 3.2. *If the elements* $a_n = r_n e^{i\theta_n}$ *are such that*

$$r_{2n-1} \geqq 2(k + \epsilon_1 - \cos\theta_{2n-1}), \qquad r_{2n} \leqq k - \epsilon \qquad (n = 1, 2, \cdots),$$

the continued fraction (1.1) *converges at least in the wider sense.*

We proceed with the proof of the following result.

COROLLARY 3.3. *If the elements* a_{2n} *lie in the circle* $|z| \leqq 1$ *and if the quantities* a_{2n+1} *are outside of and bounded away from the cardioid* $r = 2(1-\cos\theta)$ *the continued fraction* (1.1) *converges.*

To prove this one observes that one can choose numbers $k>1$ and $\epsilon_1>0$ such that the numbers a_{2n+1} lie outside the limaçon $r=2(k+\epsilon_1-\cos\theta)$. A positive number ϵ can be chosen so that $|a_{2n}| \leqq k-\epsilon$. The conditions of Theorem 3.1 are then fulfilled.

Bibliography

1. P. A. Montel, *Leçons sur les familles normales de fonctions analytique et leurs applications*, Paris, 1927.

2. Walter Leighton and H. S. Wall, *On the transformation and convergence of continued fractions*, Amer. J. Math. vol. 58 (1936) pp. 267–281.

3. Walter Leighton, *Convergence theorems for continued fractions*, Duke Math. J. vol. 5 (1939) pp. 298–308.

4. W. T. Scott and H. S. Wall, *Value regions for continued fractions*, Bull. Amer. Math. Soc. vol. 47 (1941) pp. 580–585.

5. Walter Leighton and W. J. Thron, *Continued fractions with complex elements*, Duke Math. J. vol. 9 (1942) pp. 763–772.

6. W. J. Thron, *Two families of twin convergence regions for continued fractions*, Duke Math. J. vol. 10 (1943) pp. 677–685.

The Rice Institute

EQUICONVERGENCE THEOREMS FOR ORTHONORMAL POLYNOMIALS

G. E. ALBERT AND L. H. MILLER

1. **Introduction.** If $\rho(x)$ is a non-negative function integrable on the finite interval (a, b) and positive on a set of positive measure, there exists a unique set of polynomials $\{p_n(\rho, x)\}$ of degree n, $p_n(\rho, x) = c_n x^n + \cdots$, $c_n > 0$, which are orthonormal on (a, b) relative to the weight function $\rho(x)$; that is

$$\int_a^b \rho(x) p_n(\rho, x) p_m(\rho, x) dx = \begin{cases} 0 & \text{if } n \neq m, \\ 1 & \text{if } m = n. \end{cases}$$

In the sequel the abbreviation ONP will be used to denote such a set of polynomials.

Given any function $f(x)$ for which the integral $\int_a^b \rho(x) f(x) dx$ exists, the set of ONP associated with $\rho(x)$ may be used to construct the formal expansion

$$(1) \qquad f(x) \sim \sum_{n=0}^{\infty} a_n p_n(\rho, x), \qquad a_n = \int_a^b \rho(t) f(t) p_n(\rho, t) dt.$$

Much attention has been given to the convergence properties of such series for particular choices of $\rho(x)$. In the succeeding pages known results on this problem are extended by means of what seems to be a new type of proof.

Let $\{p_n(\rho_1, x)\}$ and $\{p_n(\rho_2, x)\}$ be two sets of ONP with different weight functions $\rho_1(x)$ and $\rho_2(x)$ and let

$$(2) \qquad s_n(f; \rho_i, x) = \sum_{k=0}^{n} a_{ki} p_k(\rho_i, x), \qquad i = 1, 2,$$

denote the partial sums of nth degree for the two corresponding expansions (1) associated with $f(x)$. In §§5 and 6 below certain sufficient conditions will be established for the validity of the relation

$$(3) \qquad \lim_{n \to \infty} \{s_n(f; \rho_1, x) - s_n(f; \rho_2, x)\} = 0;$$

that is, conditions under which the two series mentioned are *equiconvergent*. It should be stated that the emphasis of the paper is upon method rather than upon specific conditions. The discerning reader

Presented to the Society, April 24, 1943; received by the editors July 9, 1943.

will easily see arrangements for the weight functions which yield variations of the conditions cited.

The materials from the theory of ONP that are needed in the proofs to follow are quite elementary and are collected for reference in §2. Two lemmas are the subject matter of §3; from them the remainder of the paper follows essentially as a series of corollaries.

Some interesting relations between bounds for related systems of ONP are indicated in §4. These are obtained as by-products of the method of proof for the equiconvergence theorems.

2. **Notations and formulas.** The symbol (a, b) will denote the closed interval $a \leqq x \leqq b$. It will be assumed without further mention that any weight function and any function $f(x)$ for which a series of type (1) is used will satisfy the conditions set forth for such functions in the introduction. The symbol $L^2(\rho; a, b)$ will denote the class of all measurable functions $f(x)$ for which the integral $\int_a^b \rho(x)[f(x)]^2 dx$ exists.

If $f(x)$ and $g(x)$ are in $L^2(\rho; a, b)$, it is well known that

$$\int_a^b \rho(x)[f(x)]^2 dx = \sum_{k=0}^{\infty} a_k^2, \tag{4}$$

$$\int_a^b \rho(x)f(x)g(x)dx = \sum_{k=0}^{\infty} a_k b_k, \qquad b_k = \int_a^b \rho(t)g(t)p_k(\rho, t)dt, \tag{5}$$

and

$$\int_a^b \rho(x)[f(x) - s_n(f; \rho, x)]^2 dx \leqq \int_a^b \rho(x)[f(x) - \pi_n(x)]^2 dx \tag{6}$$

for an arbitrary polynomial $\pi_n(x)$ of degree n.

Also

$$s_n(f; \rho, x) = \int_a^b \rho(t)f(t)K_n(\rho; x, t)dt \tag{7}$$

where

$$\begin{aligned} K_n(\rho; x, t) &= \sum_{k=0}^{n} p_k(\rho, x)p_k(\rho, t) \\ &= \frac{c_n}{c_{n+1}} \frac{p_{n+1}(\rho, t)p_n(\rho, x) - p_{n+1}(\rho, x)p_n(\rho, t)}{t - x}. \end{aligned} \tag{8}$$

In (8), c_n is the leading coefficient of $p_n(\rho, x)$ and one has

$$0 < c_n/c_{n+1} \leqq C = \text{max of } |a| \text{ and } |b|. \tag{9}$$

The reader interested in proofs of the above formulas should consult Jackson [1][1] or Szegö [4].

3. **Two inequalities.** The remaining sections of the paper will be devoted to comparisons of the two sets of ONP, $\{p_n(\rho_1, x)\}$ and $\{p_n(\rho_2, x)\}$, for which the weight functions will be assumed to have the forms

$$\rho_1(x) = \rho(x)[\pi_m(x)]^2 \quad \text{and} \quad \rho_2(x) = \rho(x)\pi_q(x)\sigma(x) \tag{10}$$

or special cases thereof. The following blanket hypotheses will be made concerning the individual factors involved in (10).

(a) *The functions $\pi_m(x)$ and $\pi_q(x)$ are polynomials of degrees m and q respectively and $\rho(x)$, $\pi_q(x)$ and $\sigma(x)$ are non-negative on (a, b).*

(b) *The polynomial $\pi_q(x)$ and the function $\sigma(x)$ are such that the product $\pi_q(x)\sigma(x)$ is measurable and bounded on (a, b) by a constant M_1.*

(c) *The polynomial $\pi_m(x)$ and the function $\sigma(x)$ are such that the quotient $\pi_m(x)/\sigma(x)$ is measurable and bounded on (a, b) by a constant M_2.*

LEMMA 1. *Let* (a), (b) *and* (c) *be satisfied. If, at a point ξ in (a, b), there exist two positive numbers $\epsilon(\xi)$ and $\gamma(\xi)$ such that when x is in the common part of the intervals (a, b) and $(\xi-\epsilon, \xi+\epsilon)$ the condition*

$$\left|\frac{\pi_m(\xi)}{\sigma(\xi)} - \frac{\pi_m(x)}{\sigma(x)}\right| < \gamma(\xi)\cdot|x-\xi| \tag{11}$$

holds, then at the point ξ

$$\begin{aligned}\left\{\int_a^b \rho(t)\pi_q(t)\sigma(t)\left[\frac{\{\pi_m(t)\}^2}{\sigma(t)}K_{n+q}(\rho_1;\xi,t) - \pi_q(\xi)K_k(\rho_2;\xi,t)\right]^2 dt\right\}^{1/2} \\ \leqq M_1^{1/2}M_2\left\{\sum_{j=k-m+1}^{n+q}[p_j(\rho_1,\xi)]^2\right\}^{1/2} \\ + M_1^{1/2}C\left\{\left(\frac{4M_2^2}{\epsilon^2}+\gamma^2\right)\sum_{j=k-m}^{k-m+1}[p_j(\rho_1,\xi)]^2\right\}^{1/2}\end{aligned} \tag{12}$$

for all k and n such that $m \leqq k \leqq n$. The constant C is the bound indicated in (9).

PROOF. Since $\pi_q(\xi)p_j(\rho_2, \xi)$ is a polynomial of degree $j+q$ in ξ, one has

---

[1] The numbers in brackets refer to the bibliography.

$$\pi_q(\xi)p_j(\rho_2, \xi) = \int_a^b \rho_1(t)\pi_q(t)p_j(\rho_2, t)K_{n+q}(\rho_1; \xi, t)dt$$

(13)

$$= \int_a^b \rho_2(t)\left[\frac{\{\pi_m(t)\}^2}{\sigma(t)} K_{n+q}(\rho_1; \xi, t)\right] p_j(\rho_2, t)dt$$

for all $j \leqq n$. This implies

$$\pi_q(\xi)K_k(\rho_2; \xi, t) \equiv s_k\left(\left[\frac{\pi_m^2}{\sigma} K_{n+q}(\rho_1)\right]; \rho_2, t\right).$$

In view of (6), the left member of (12) is dominated by

$$\left\{\int_a^b \rho_2(t)\left[\frac{\{\pi_m(t)\}^2}{\sigma(t)} K_{n+q}(\rho_1; \xi, t) - \frac{\pi_m(\xi)\pi_m(t)}{\sigma(\xi)} K_{k-m}(\rho_1; \xi, t)\right]^2 dt\right\}^{1/2}$$

(14)

$$\leqq \left\{M_1M_2^2\int_a^b \rho_1(t)\left[\sum_{j=k-m+1}^{n+q} p_j(\rho_1, \xi)p_j(\rho_1, t)\right]^2 dt\right\}^{1/2}$$

$$+ \left\{M_1\int_a^b \rho_1(t)\left[\frac{\pi_m(t)}{\sigma(t)} - \frac{\pi_m(\xi)}{\sigma(\xi)}\right]^2 [K_{k-m}(\rho_1; \xi, t)]^2 dt\right\}^{1/2}.$$

The first term in (14) is clearly bounded by the first term in (12) on the right. In the second term of (14) the range of integration must be broken into the three intervals $(a, \xi-\epsilon)$, $(\xi-\epsilon, \xi+\epsilon)$ and $(\xi+\epsilon, b)$. Using (8), (9), condition (c), and the inequality $|\xi-x| \geqq \epsilon(\xi)$, the contribution to the integral in that term from the first and third subintervals is found to be dominated by

$$\frac{4M_1M_2^2C^2}{\epsilon^2}\int_a^b \rho_1(t)[p_{k-m+1}(\rho_1, \xi)p_{k-m}(\rho_1, t) - p_{k-m+1}(\rho_1, t)p_{k-m}(\rho_1, \xi)]^2 dt$$

$$\leqq \frac{4M_1M_2^2C^2}{\epsilon^2}[\{p_{k-m}(\rho_1, \xi)\}^2 + \{p_{k-m+1}(\rho_1, \xi)\}^2].$$

Use of (8), (9) and (11) shows that the contribution to the integral under consideration from $(\xi-\epsilon, \xi+\epsilon)$ is dominated by

$$M_1C^2\gamma^2[\{p_{k-m}(\rho_1, \xi)\}^2 + \{p_{k-m+1}(\rho_1, \xi)\}^2].$$

Combining all these results one has (12).

The following special case of Lemma 1 is of special importance in the applications.

LEMMA 2. *Let the hypotheses* (a) *and* (b) *be satisfied. If the quotient* $\pi_m(x)/\sigma(x)$ *satisfies the Lipschitz condition*

$$\left|\frac{\pi_m(x)}{\sigma(x)} - \frac{\pi_m(t)}{\sigma(t)}\right| < \lambda |x - t| \tag{15}$$

for all x and t in (a, b), then, for every point ξ in (a, b),

$$\begin{aligned}\left\{\int_a^b \rho_2(t)\left[\frac{\{\pi_m(t)\}^2}{\sigma(t)} K_{n+q}(\rho_1; \xi, t) - \pi_q(\xi) K_k(\rho_2; \xi, t)\right]^2 dt\right\}^{1/2} \\ \leqq M_1^{1/2} M_2 \left\{\sum_{j=k-m+1}^{n+q} [p_j(\rho_1, \xi)]^2\right\}^{1/2} \\ + \lambda M_1^{1/2} C \left\{\sum_{j=k-m}^{k-m+1} [p_j(\rho_1, \xi)]^2\right\}^{1/2}\end{aligned} \tag{16}$$

for all k and n such that $m \leqq k \leqq n$. The constant C is the bound indicated in (9).

PROOF. The hypothesis (c) is automatically satisfied. The proof proceeds like that for Lemma 1 except that (8), (9) and the Lipschitz condition are to be applied directly to the second term in (14) to obtain the second term in (16).

4. **Bounds for ONP.** Lemmas 1 and 2 are powerful tools for establishing bounds for ONP. Since the most interesting bounds are uniform in x on some subset of (a, b), attention will be focused upon the application of Lemma 2; extensions of the results which are obtainable from Lemma 1 are left to the reader.

THEOREM 1. *Suppose the hypotheses* (a) *and* (b) *and the Lipschitz condition* (15) *are satisfied. If at a point ξ in (a, b) for which $\pi_q(\xi)$ is different from zero there is a number $H_1(\xi)$ such that $|p_n(\rho_1, \xi)| \leqq H_1$ for all n, then there exists a number $H_2(\xi)$ such that $|p_n(\rho_2, \xi)| \leqq H_2$ for all n. Moreover, if for every point of some closed subset of (a, b) on which $\pi_q(x)$ is different from zero the bound H_1 exists and is independent of x, then H_2 exists and is independent of x on that set.*

PROOF. Obviously

$$0 \equiv \int_a^b \rho_2(t) p_n(\rho_2, t) \pi_q(\xi) K_{n-1}(\rho_2; \xi, t) dt.$$

Combining this with (13) for $j = n$, one has

$$\begin{aligned}&\pi_q(\xi) p_n(\rho_2, \xi) \\ &= \int_a^b \rho_2(t) p_n(\rho_2, t)\left[\frac{\{\pi_m(t)\}^2}{\sigma(t)} K_{n+q}(\rho_1; \xi, t) - \pi_q(\xi) K_{n-1}(\rho_2; \xi, t)\right] dt.\end{aligned}$$

Using Schwarz' inequality, one obtains

$$|\pi_q(\xi)p_n(\rho_2,\xi)| \leqq \left\{\int_a^b \rho_2(t)[p_n(\rho_2,t)]^2dt \cdot \int_a^b \rho_2(t)\left[\frac{\{\pi_m(t)\}^2}{\sigma(t)}K_{n+q}(\rho_1;\xi,t) - \pi_q(\xi)K_{n-1}(\rho_2;\xi,t)\right]^2 dt\right\}^{1/2}.$$

The first factor on the right is unity; the second factor can be treated by Lemma 2. This gives, for all $n>m$,

$$|\pi_q(\xi)p_n(\rho_2,\xi)| \leqq M_1^{1/2}M_2\left\{\sum_{j=n-m}^{n+q}[p_j(\rho_1,\xi)]^2\right\}^{1/2} + \lambda M_1^{1/2}C\left\{\sum_{j=n-m-1}^{n-m}[p_j(\rho_1,\xi)]^2\right\}^{1/2}.$$

The theorem follows since the number of terms on the right is independent of n.

Theorem 1 is a generalization of a theorem due to Korous. See Jackson [1, p. 205] or Szegö [4, p. 157]. Results of a similar character have also been established by Peebles [2].

Remark. One or two examples showing the disposition of the factors in (10) may be helpful. Setting $\sigma(x)\equiv\pi_m(x)\equiv 1$ in Theorem 1 refers the boundedness of the ONP of weight function $\rho(x)\pi_q(x)$ to that of the ONP of weight function $\rho(x)$. This result will be used in §5.

The normalized Jacobi polynomials have the weight function $(1-x)^\alpha(1+x)^\beta$ where $\alpha>-1$ and $\beta>-1$; the interval (a, b) is $(-1, 1)$. They can be compared directly with the polynomials $p_n(x)=(2/\pi)^{1/2}\cos\theta$, $x=\cos\theta$, whose weight function is $(1-x^2)^{-1/2}$ by the following:

$$\rho(x) = (1-x)^{-2m_1-1/2}(1+x)^{-2m_2-1/2},$$

$$\sigma(x) = (1-x)^{\alpha+2m_1-q_1+1/2}(1+x)^{\beta+2m_2-q_2+1/2},$$

$$\pi_q(x) = (1-x)^{q_1}(1+x)^{q_2} \quad \text{and} \quad \pi_m(x) = (1-x)^{m_1}(1+x)^{m_2},$$

where m_1, m_2 are the smallest integers such that $m_1\geqq-\alpha/2-1/4$, and $m_2\geqq-\beta/2-1/4$ and q_1, q_2 are the smallest integers such that $q_1\geqq\alpha+m_1+3/2$ and $q_2\geqq\beta+m_2+3/2$. Then in Lemma 2, $\rho\pi_q\sigma=(1-x)^\alpha(1+x)^\beta$ and $\rho[\pi_m]^2=(1-x^2)^{-1/2}$ and it is easy to show that the hypotheses are satisfied.

It is fundamental in such applications that the factor $\rho(x)$ *need not be integrable* for the proofs of Lemmas 1 and 2. One merely makes sure of the integrability of the products $\rho_1(x)$, $\rho_2(x)$, $\pi_q(x)\sigma(x)$, and the quotient $\pi_m(x)/\sigma(x)$.

5. Equiconvergence theorems. Walsh and Wiener [5] have given a necessary and sufficient condition for equiconvergence of two general orthogonal series. Their sufficiency condition will be given below in a form suited to the study of ONP.

LEMMA 3. *If $\sigma(x)$ is in $L^2(\rho; a, b)$ and $\{p_n(\rho, x)\}$ and $\{p_n(\rho\sigma, x)\}$ are the ONP with weight functions $\rho(x)$ and $\rho(x)\sigma(x)$, a sufficient condition for*

$$\lim_{n\to\infty} \{s_n(f; \rho, \xi) - s_n(f; \rho\sigma, \xi)\} = 0, \qquad a \leqq \xi \leqq b, \tag{17}$$

for all $f(x)$ in $L^2(\rho; a, b)$ is that there exist two numbers $N(\xi)$ and $A(\xi)$ such that

$$\int_a^b \rho(t)[K_n(\rho; \xi, t) - \sigma(t)K_n(\rho\sigma; \xi, t)]^2 dt \leqq A(\xi) \tag{18}$$

for all $n \geqq N$. Moreover, if N and A are independent of ξ on some subset S of (a, b), then (17) holds uniformly on S.

PROOF. For an arbitrary point ξ in (a, b) let

$$a_k = \int_a^b \rho(t)f(t)p_k(\rho, t)dt,$$

$$b_k(\xi) = \int_a^b \rho(t)[K_n(\rho; \xi, t) - \sigma(t)K_n(\rho\sigma; \xi, t)]p_k(\rho, t)dt.$$

Now $b_k(\xi) = p_k(\rho, \xi) - p_k(\rho, \xi) = 0$ for all $k \leqq n$. Thus by (5)

$$s_n(f; \rho, \xi) - s_n(f; \rho\sigma, \xi) = \int_a^b \rho(t)f(t)[K_n(\rho; \xi, t) - \sigma(t)K_n(\rho\sigma; \xi, t)]dt$$

$$= \sum_{k=n+1}^{\infty} a_k b_k(\xi).$$

But then $|s_n(f; \rho, \xi) - s_n(f; \rho\sigma, \xi)|$ is bounded by

$$\left[\sum_{k=n+1}^{\infty} a_k^2\right]^{1/2}\left[\sum_{k=0}^{\infty} \{b_k(\xi)\}^2\right]^{1/2}$$

$$\leqq \left[\sum_{k=n+1}^{\infty} a_k^2\right]^{1/2}\left[\int_a^b \rho(t)[K_n(\rho; \xi, t) - \sigma(t)K_n(\rho\sigma; \xi, t)]^2 dt\right]^{1/2},$$

where (4) has been used to obtain the inequality. The limit (17) follows at once on using (18) and the well known fact that $\lim_{n\to\infty} \sum_{k=n+1}^{\infty} a_k^2 = 0$. The uniformity clause is evident.

THEOREM 2. *If $\rho(x)$ and $\sigma(x)$ are non-negative and $\sigma(x)$ is bounded and measurable on (a, b) and a polynomial $\pi_m(x)$ of degree m can be found such that the quotient $\pi_m(x)/\sigma(x)$ satisfies a Lipschitz condition, then* (17) *holds, for all $f(x)$ in $L^2(\rho; a, b)$, at any point ξ in (a, b) at which $\pi_m(\xi)$ is different from zero and $|p_n(\rho, \xi)| \leqq H(\xi)$ where $H(\xi)$ is a number independent of n. Moreover,* (17) *holds uniformly on any closed subset of (a, b) on which $\pi_m(x)$ is different from zero and the number H is independent of x.*

PROOF. To demonstrate the inequality (18) set

$$I_n(\xi) = \int_a^b \rho(t)[K_n(\rho; \xi, t) - \sigma(t)K_n(\rho\sigma; \xi, t)]^2 dt.$$

Add and subtract $[\pi_m(t)]^2 K_n(\rho\pi_m^2; \xi, t)$ inside the bracket in the integrand. Here $K_n(\rho\pi_m^2; \xi, t)$ denotes the kernel polynomial (8) for the ONP of weight function $\rho(x)[\pi_m(x)]^2$. Applying Minkowski's inequality and a little obvious manipulation, one has

$$[I_n(\xi)]^{1/2} \leqq \left[M_1 \int_a^b \rho(t)\sigma(t)\left[\frac{\{\pi_m(t)\}^2}{\sigma(t)} K_n(\rho\pi_m^2; \xi, t) - K_n(\rho\sigma; \xi, t)\right]^2 dt\right]^{1/2} + \left[\int_a^b \rho(t)\left[\{\pi_m(t)\}^2 K_n(\rho\pi_m^2; \xi, t) - K_n(\rho; \xi, t)\right]^2 dt\right]^{1/2},$$

where M_1 is the bound on $\sigma(x)$. To the first term on the right apply Lemma 2 with $\pi_q(x) \equiv 1$, $k = n$; to the second term apply Lemma 2 with $\pi_q(x) \equiv \sigma(x) \equiv 1$ and $k = n$. The result is

$$[I_n(\xi)]^{1/2} \leqq (M_1M_2 + M_3)\left[\sum_{j=n-m+1}^{n} \{p_j(\rho\pi_m^2, \xi)\}^2\right]^{1/2} + \lambda C(M_1 + 1)\left[\sum_{j=n-m}^{n-m+1} \{p_j(\rho\pi_m^2, \xi)\}^2\right]^{1/2}$$

for all $n \geqq m$. The constants M_2 and M_3 are bounds on π_m/σ and π_m. By the remark following Theorem 1 the system $\{p_n(\rho\pi_m^2, \xi)\}$ is bounded as to n at any point ξ such that $\pi_m(\xi)$ is different from zero and at which the system $\{p_n(\rho, \xi)\}$ is so bounded. This proves the inequality (18) under the stated hypotheses. The uniformity clause is evident.

The following more general theorem is obtained by applying Lemma 1 to the integral $I_n(\xi)$.

THEOREM 3. *Let $\rho(x)$ and $\sigma(x)$ be non-negative and $\sigma(x)$ be bounded and measurable on (a, b). Let ξ be a point in (a, b). If there is a polynomial $\pi_m(x)$ of degree m such that $\pi_m(\xi)$ is different from zero and $\pi_m(x)/\sigma(x)$ satisfies the condition* (11) *of Lemma* 1 *and if there is a number $H(\xi)$ independent of n such that $|p_n(\rho, \xi)| \leqq H(\xi)$ for all n, then* (17) *holds for all $f(x)$ in $L^2(\rho; a, b)$.*

Theorems 2 and 3 furnish no information about equiconvergence at a point where the auxiliary polynomial $\pi_m(x)$ vanishes. It is not difficult to construct examples which show that such equiconvergence *cannot be expected* for all functions in $L^2(\rho; a, b)$.

6. **Applications.** Let (a, b) be $(-1, 1)$ and let $\rho(x)$ and $\sigma(x)$ have the forms

$$\rho(x) = (1 - x^2)^{-1/2} \quad \text{and} \quad \sigma(x) = \tau(x) \prod_{k=1}^{h} |x - x_k|^{\lambda_k}$$

where $-1 \leqq x_1 < x_2 < \cdots < x_h \leqq 1$; $\lambda_k \geqq 0$, $k = 1, 2, \cdots, h$, and $0 < A \leqq \tau(x) \leqq B$. Theorem 2 applies if $\tau(x)$ satisfies a Lipschitz condition on $(-1, 1)$ and Theorem 3 applies at any point ξ in $(-1, 1)$ distinct from all the points x_k, $k = 1, 2, \cdots, h$, provided that there is a pair of numbers $\lambda(\xi)$ and $\epsilon(\xi, \lambda)$ such that

$$|\tau(x) - \tau(\xi)| < \lambda |x - \xi|$$

for all x in $(-1, 1)$ such that $|x - \xi| < \epsilon$. In either case the polynomial $\pi_m(x)$ is chosen to be $\prod_{k=1}^{h} (x - x_k)^{m_k}$, where m_k is the smallest integer such that $m_k - \lambda_k \geqq 1$. These results compare the convergence of the series (1) for the ONP of weight function $\rho(x)\sigma(x)$ with the convergence of the Fourier cosine series on the open intervals $x_i < x < x_{i+1}$, $i = 1, 2, \cdots, h$. Either case includes the Jacobi polynomials of weight function $(1-x)^\alpha(1+x)^\beta$ where $\alpha, \beta > -1/2$. The cases where α or β or both are less than $-1/2$ are also easily handled. For example, if $\alpha = \beta = -3/4$, set $\rho(x) = (1-x^2)^{-3/4}$ and $\sigma(x) = (1-x^2)^{1/4}$ and choose $\pi_m(x)$ appropriately.

The case above using Theorem 3 essentially includes a theorem of Szegö [4; Theorem 13.1.2, p. 307]. The case using Theorem 2 is comparable to results of Peebles [2]. The methods of the present paper are certainly more elementary and more direct than those used by either of the above authors.

BIBLIOGRAPHY

1. D. Jackson, *Fourier series and orthogonal polynomials*, 1941.

2. G. H. Peebles, *An equivalence theorem for series of orthogonal polynomials*, Proc. Nat. Acad. Sci. U.S.A. vol. 25 (1939) p. 97.

3. ———, *On equivalence of certain types of series of orthonormal functions*, Bull. Amer. Math. Soc. vol. 48 (1942) pp. 556–561.

4. G. Szegö, *Orthogonal polynomials*, New York, 1939.

5. J. L. Walsh and N. Wiener, *The equivalence of expansions in terms of orthogonal functions*, Journal of Mathematics and Physics, Massachusetts Institute of Technology vol. 1 (1922) p. 103.

OHIO STATE UNIVERSITY

MATRIX PRODUCTS OF MATRIX POWERS

R. F. CLIPPINGER

1. **Introduction.** Let m n-by-n matrices, A_k, of complex constants, a_{ijk} $(i, j=1, 2, \cdots, n; k=1, 2, \cdots, m)$, be given. We shall denote by $\mathfrak{L}$ the set of all matrices,

$$A(t) = \sum_{i=1}^{m} \rho_i(t) A_i,$$

where $\rho_i(t)$ $(i=1, 2, \cdots, m)$ are arbitrary, non-negative, summable functions of the real variable t on the interval T, $a \leqq t \leqq b$. We shall call $\mathfrak{I}$, $\mathfrak{S}$, or $\mathfrak{X}$ the subsets of $\mathfrak{L}$ obtained by restricting the functions $\rho_i(t)$ to polynomial functions, step functions, or step functions which are all zero except one. Since, in each case, the elements of $A(t)$ are summable functions of t on T, it follows that, on T, there exists a unique, absolutely continuous matrix solution,[1] $Y(t)$, of the linear, matrix differential equation and initial condition:

$$dY(t)/dt = Y(t)A(t), \qquad Y(a) = E, \tag{1.1}$$

where E is the n-by-n unit matrix. We shall denote by λ, ι, σ or ξ the set of matrices, $Y(t)$, which are particular values of solutions of (1.1), where $A(t)$ is an arbitrary matrix of $\mathfrak{L}$, $\mathfrak{I}$, $\mathfrak{S}$, or $\mathfrak{X}$, respectively, and t is on T.

If A is a matrix with elements a_{ij}, let the absolute value of A and the exponential and natural logarithm of A be defined[2] by the equations:

$$|A| = \left[\sum_{i,j=1}^{n} |a_{ij}|^2\right]^{1/2},$$

$$\exp A = \sum_{i=0}^{\infty} A^i/i!$$

$$\log A = \sum_{i=1}^{\infty} (-1)^{i-1}(A - E)^i/i, \quad \text{if} \quad |A - E| < 1.$$

Presented to the Society, April 18, 1942; received by the editors April 4, 1943, and, in revised form, August 18, 1943. The author wishes to thank Professor G. D. Birkhoff for suggestions which led to this paper.

[1] See W. M. Whyburn, *On the fundamental existence theorems for differential systems.* Ann. of Math. (2) vol. 30 (1928–29) p. 31. We observe that equations (1.1) are equivalent to a system of $2n$ real, linear, first order differential equations satisfying all the hypotheses of this theorem.

[2] See J. v. Neumann, *Über die analytischen Eigenschaften von Gruppen linearer Transformationen und ihrer Darstellungen.* Math. Zeit. vol. 30 (1929) pp. 6, 7.

J. v. Neumann[3] has shown that if j is an integer and if $|A-E|<1$, then

$$A^j = \exp [j \log A]. \tag{1.2}$$

Hence, for any α, we define the α power function of A by the equation

$$A^\alpha = \exp (\alpha \log A), \quad \text{if} \quad |A - E| < 1.$$

If $|\exp A_i - E| < 1$ $(i = 1, 2, \cdots, m)$, we define μ as the set of matrix products of matrix powers,

$$\prod_{j=1}^{J} \prod_{i=1}^{m_i} (\exp A_i)^{\alpha_{ij}},$$

where the α_{ij} are arbitrary non-negative numbers.

Let us identify an n-by-n matrix, B, of complex numbers with the point in $2n^2$-Euclidean space, whose coordinates are the real and imaginary parts of the elements of B. The distance between two points B_1 and B_2 may be defined as $|B_1 - B_2|$. A set, β, of matrices, B, is then also a point set whose closure we denote by $\bar{\beta}$.

It is the purpose of this paper to show that the sets $\bar{\lambda}$, $\bar{\iota}$, $\bar{\sigma}$, $\bar{\xi}$ and, if it exists, $\bar{\mu}$, are identical.

2. Principal theorems.

THEOREM 2.1. *The sets $\bar{\lambda}$ and $\bar{\sigma}$, defined above, are identical.*

Since any step function on T is summable on T, it follows that $\sigma \subset \lambda$. Suppose $A_L(t)$ is a matrix of class $\mathcal{L}$ with coefficients $\rho_i(t)$. Then, for all positive δ, there exists a matrix, A_S, of class S with coefficients $r_i(t)$ such that

$$\sum_{k=1}^{m} \int_a^b |\rho_k(t) - r_k(t)|\,|a_{ijk}|\,dt < \delta \qquad (i, j = 1, 2, \cdots, n).$$

Let the corresponding solutions of (1.1) be $Y_L(t)$ and $Y_S(t)$. Since $Y(t)$ is a uniformly continuous functional[4] of $A(t)$, it follows that, given any positive number ϵ, δ may be so chosen that

$$Y_L(t) - Y_S(t) \ll \epsilon;$$

that is, the absolute value of each element of the matrix on the left is less than ϵ. Hence $\lambda \subset \bar{\sigma}$.

[3] Loc. cit. pp. 8, 12.

[4] By this we mean that the elements of $Y(t)$ are uniformly continuous functionals of the elements of $A(t)$. See W. M. Whyburn, *Functional properties of the solutions of differential systems*. Trans. Amer. Math. Soc. vol. 32 (1930) p. 508.

THEOREM 2.2. *The sets $\bar{\sigma}$ and ξ are identical.*

Clearly $\xi \subset \sigma$.

J. v. Neumann[5] has shown that if $|A|, |B| < (1/2) \log (3/2)$, then

$$\log [\exp A \exp B] = A + B + O(|A||B|).$$

This equation, by induction, leads to the generalized equation:

$$\log \prod_{i=1}^{m} \exp A_i = \sum_{i=1}^{m} A_i + \sum_{i,j=1}^{m} O(|A_i||A_j|), \tag{2.1}$$

if $|A_i| < \delta(m)$, where it suffices to take $\delta(m) < [\log (3/2)]/2n(m-1)$. To this we may add the equation,

$$\exp (A + B) = \exp A + O(|B|) \tag{2.2}$$

which follows immediately from the definition of $\exp (A+B)$.

LEMMA 2.1. *If A is a matrix of constants, the matrix,*

$$Y(t) = Y_0 \exp [(t - a)A],$$

is the solution of the linear, matrix differential equation and initial condition

$$dY(t)/dt = Y(t)A, \qquad Y(a) = Y_0.$$

The series $\sum_{j=0}^{\infty} (t-a)^j A^j/j$ is uniformly convergent[6] on any interval $|t-a| < N$, hence the lemma may be established by term-by-term differentiation.

LEMMA 2.2. *If $A(t)$ is a matrix of summable functions, and if $A(t) \ll (M)$ on T, the solution, $Y(t)$, of equation (1.1) satisfies the inequality*

$$Y(t) - E \ll (1/n[\exp Mn(t - a) - 1]) \text{ on } T.$$

Slight modifications of the proof of the existence theorem given by G. D. Birkhoff and R. E. Langer[7] yield this lemma.

LEMMA 2.3. *If $B_1, B_2, \cdots, B_m$ are square matrices,*

$$\lim_{j \to \infty} \left[\prod_{i=1}^{m} \exp (B_i/j) \right]^j = \exp \sum_{i=1}^{m} B_i.$$

---

[5] Loc. cit. pp. 13–15.

[6] See J. v. Neumann, loc. cit. p. 7.

[7] *The boundary problems and developments associated with a system of ordinary linear differential equations of the first order.* Proceedings of the American Academy of Arts and Sciences vol. 58 (1922–1923) pp. 59–63.

Let $P_j=\prod_{i=1}^{m}\exp(B_i/j)$. By Lemma 2.1, P_j is the solution for $t=1$ of the differential equation

$$dY(t)/dt = Y(t)\sum_{i=1}^{m}\rho_i(t)B_i, \qquad Y(0) = E,$$

where the $\rho_i(t)$ are all zero except on the interval $(i-1)/m \leqq t \leqq i/m$ where $\rho_i(t)=m/j$ $(i=1, 2, \cdots, m)$. Let B be an upper bound to the absolute values of the elements of mB_i. Then $\sum_{i=1}^{m}\rho_i(t)B_i \ll (B/j)$. By Lemma 2.2, j may be chosen so large that $|P_j-E|<1$ and $|B_i/j|<[\log(3/2)]/2n(m-1)$. Hence by (1.2)

$$P_j = \exp\left[j\log\prod_{i=1}^{m}\exp(B_i/j)\right].$$

From (2.1) it follows that

$$j\log\prod_{i=1}^{m}\exp(B_i/j) = \sum_{i=1}^{m}B_i + O(1/j);$$

and, finally, (2.2) implies that

$$P_j = \exp\sum_{i=1}^{m}B_i + O(1/j).$$

This establishes Lemma 2.3.

LEMMA 2.4. *If each of K n-by-n matrices, C_k, is the limit of a sequence, $\{C_{kj}\}$, of matrices, then*

$$\lim_{j\to\infty}\prod_{k=1}^{K}C_{kj} = \prod_{k=1}^{K}C_k.$$

This lemma may be established easily by induction.

To prove that $\sigma\subset\xi$, let Y_S be a matrix of σ; then Y_S is a product of a finite number of matrices of the form

$$C_k = \exp\sum_{i=1}^{m}\alpha_{ik}A_i,$$

and by Lemma 2.3, each C_k is a limit of matrices

$$C_{kj} = \left[\prod_{i=1}^{m}\exp(\alpha_{ik}A_i/j)\right]^j.$$

Hence, by Lemma 2.4, Y_S is a limit of matrices of ξ and therefore Y_S is a member of ξ.

COROLLARY TO THEOREM 2.2. *If* $|\exp A_i - E| < 1$ $(i = 1, 2, \cdots, m)$, *then* $\xi = \mu$.

To prove this, we need only observe that, by definition,

$$(\exp A_i)^{a_{ik}/j} = \exp (a_{ik}A_i/j) \quad \text{if} \quad |\exp A_i - E| < 1.$$

THEOREM 2.3. *The sets* $\bar{\lambda}$ *and* $\bar{\iota}$ *are identical.*

Since $\mathfrak{J} \subset \mathcal{L}$, $\iota \subset \lambda$. Given a matrix in $\mathcal{L}$ with coefficients $\rho_i(t)$, there exists a matrix in $\mathfrak{J}$, with coefficients $\mu_i(t)$, such that

$$\sum_{k=1}^{m} \int_a^b |\rho_k(t) - \mu_k(t)|\,|a_{ijk}|\,dt$$

is arbitrarily small $(i, j = 1, 2, \cdots, n)$. Therefore[8] $\iota \subset \bar{\lambda}$.

CARNEGIE INSTITUTE OF TECHNOLOGY

---

[8] See W. M. Whyburn, *Functional properties of the solutions of differential systems* Loc. cit.

ON THE UNIQUENESS OF YOUNG'S DIFFERENTIALS

CHENG-NING YANG

Introduction. The differential of a function of several variables may be defined in a variety of ways, of which the one given by Young[1] renders the best parallelism with the case of a single variable. Stated in the way given below, his definition is applicable to a function defined in a set S of points containing limiting points at which the function is to have differentials. The question of the uniqueness of the differentials, however, arises. In this paper we shall first define, and prove two theorems concerning, the "limiting directions" which describe the directional distribution of the points of S near a limiting point. Then we proceed to show that properties of these limiting directions determine whether the differential is unique or not.

Consider the n-dimensional space. We use the notation for the scalar product of two vectors $a(a_1, a_2, \cdots, a_n)$ and $b(b_1, b_2, \cdots, b_n)$: $a \cdot b = \sum_{i=1}^{n} a_i b_i$. By $\|a\|$ we mean $+(a \cdot a)^{1/2}$; and when $\|a\| = 1$, a is called a direction.

DEFINITION. Let α be a limiting point of S (this will be assumed throughout the paper). If l is a direction such that

$$\left\{ \text{lower bound of } \left\| \frac{x - \alpha}{\|x - \alpha\|} - l \right\| \text{ for } x \text{ in } S \text{ and } \delta > \|x - \alpha\| > 0 \right\} = 0$$

for all positive δ, it is said to be a limiting direction of S at α.

THEOREM 1. *S has at least one limiting direction at α.*

THEOREM 2. *If ω is a direction perpendicular to all of the limiting directions of S at α, then*

$$\lim_{x \text{ in } S,\ x \to \alpha} \omega \cdot \frac{x - \alpha}{\|x - \alpha\|} = 0.$$

Each of the above theorems is readily derived by means of an indirect proof, with the help of the Heine-Borel theorem in the case of Theorem 1.

Let a function $f(x)$ be defined for all x in S. Young's definition of the differential may be stated in the following way:

DEFINITION OF THE DIFFERENTIAL. If there exists a vector V such that

Received by the editors August 31, 1943.

[1] Proc. London Math. Soc. (2) vol. 7 (1908) p. 157.

$$\lim_{x \text{ in } S,\, x \to \alpha} \frac{f(x) - f(\alpha) - V \cdot (x - \alpha)}{\|x - \alpha\|} = 0,$$

$f(x)$ is said to have a differential $df(\alpha, \eta) = V \cdot \eta$ at α.

THEOREM 3. *The differential $df(\alpha, \eta)$, if it exists, is unique when S has, at α, n linearly independent limiting directions.*

PROOF. Suppose both $U \cdot \eta$ and $V \cdot \eta$ can be written as $df(\alpha, \eta)$. Let $l_1, l_2, \cdots, l_n$ be n linearly independent limiting directions of S at α. The theorem is proved if we can show that $(U - V) \cdot l_i = 0$, $i = 1, 2, \cdots, n$.

To prove this we have

$$\lim_{x \text{ in } S,\, x \to \alpha} (U - V) \cdot \frac{x - \alpha}{\|x - \alpha\|} = \lim_{x \text{ in } S,\, x \to \alpha} \frac{f(x) - f(\alpha) - V \cdot (x - \alpha)}{\|x - \alpha\|} - \lim_{x \text{ in } S,\, x \to \alpha} \frac{f(x) - f(\alpha) - U \cdot (x - \alpha)}{\|x - \alpha\|} = 0.$$

Hence given any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\left| (U - V) \cdot \frac{x - \alpha}{\|x - \alpha\|} \right| < \epsilon$$

for all x in S satisfying $\delta \geqq \|x - \alpha\| > 0$. Since

$$\left\{ \text{lower bound of } \left\| \frac{x - \alpha}{\|x - \alpha\|} - l_i \right\| \text{ for } \delta > \|x - \alpha\| > 0 \text{ and } x \text{ in } S \right\} = 0,$$

there must be a point x_0 in S such that

$$\delta > \|x_0 - \alpha\| > 0 \quad \text{and} \quad \left\| \frac{x_0 - \alpha}{\|x_0 - \alpha\|} - l_i \right\| < \epsilon.$$

Thus

$$\begin{aligned} |(U - V) \cdot l_i| &= \left| (U - V) \cdot \left[l_i - \frac{x_0 - \alpha}{\|x_0 - \alpha\|} + \frac{x_0 - \alpha}{\|x_0 - \alpha\|} \right] \right| \\ &\leqq \left| (U - V) \cdot \left(l_i - \frac{x_0 - \alpha}{\|x_0 - \alpha\|} \right) \right| + \left| (U - V) \cdot \frac{x_0 - \alpha}{\|x_0 - \alpha\|} \right| \\ &\leqq \|U - V\| \left\| l_i - \frac{x_0 - \alpha}{\|x_0 - \alpha\|} \right\| + \epsilon \leqq [\|U - V\| + 1]\epsilon. \end{aligned}$$

But ϵ is arbitrary, hence

$$(U - V)\cdot l_i = 0.$$

THEOREM 4. *If S has $m(<n)$ linearly independent limiting directions $l_1, l_2, \cdots, l_m$ at α, and all its other limiting directions are linear combinations of them, and if $V\cdot\eta$ is a differential of $f(x)$ at α, the most general form of the differential is*

$$\left(V + \sum_{i=1}^{n-m} K_i\omega_i\right)\cdot\eta, \tag{A}$$

where K_i are arbitrary constants and $\omega_1, \omega_2, \cdots, \omega_{n-m}$ are linearly independent directions perpendicular to the l's.

PROOF. If $U\cdot\eta$ is also a differential at α, from the proof of the last theorem we have $(U-V)\cdot l_j=0$, $j=1, 2, \cdots, m$. Thus $(U-V)$ must be a linear combination of the vectors $\omega_1, \omega_2, \cdots, \omega_{n-m}$ which are perpendicular to the l's, that is, U is of the form (A).

To show that $(V+\sum_{i=1}^{n-m}K_i\omega_i)\cdot\eta$ is actually a differential, we have by Theorem 2

$$\lim_{x \text{ in } S,\, x\to\alpha} \frac{f(x) - f(\alpha) - (V + \sum_{i=1}^{n-m}K_i\omega_i)\cdot(x-\alpha)}{\|x-\alpha\|}$$

$$= \lim_{x \text{ in } S,\, x\to\alpha} \frac{f(x) - f(\alpha) - V\cdot(x-\alpha)}{\|x-\alpha\|} - \sum_{1}^{n-m} K_i \lim_{x \text{ in } S,\, x\to\alpha} \omega_i\cdot\frac{x-\alpha}{\|x-\alpha\|} = 0.$$

The theorem follows immediately.

COROLLARY. *There exists a unique differential $U\cdot\eta$ where U is in the m-dimensional space formed by $l_1, l_2, \cdots, l_m$.*

Finally I wish to express my thanks to Professor Y. Y. Tseng without whose continual encouragement this paper would never have been completed.

NATIONAL SOUTHWEST ASSOCIATED UNIVERSITY,
KUNMING, CHINA

RESOLUTION OF TEMPERATURE PROBLEMS BY THE USE OF FINITE FOURIER TRANSFORMATIONS

HERBERT KAPFEL BROWN[1]

1. **The finite sine transformation of the convolution.** The finite sine transformation and the finite cosine transformation of $U(x)$ with respect to x are defined[2] by

$$u_s(n) = \int_0^\pi U(x) \sin nx dx = S\{U(x)\}$$

and

$$u_c(n) = \int_0^\pi U(x) \cos nx dx = C\{U(x)\},$$

respectively.[3] In particular, $S\{U'\} = -nC\{U\}$, and $S\{U''\} = -n[(-1)^n U(\pi) - U(0)] - n^2 S\{U\}$. Further, if $U(x)$ and $U'(x)$ vanish at the end points of the interval $(0, \pi)$, then $S\{U''\} = -n^2 S\{U\}$, and $C\{U'\} = nS\{U\}$.

The convolution $U_1 * U_2$, or Faltung,[2] of the two functions $U_1(x)$, $-2\pi \leqq x \leqq 2\pi$, and $U_2(x)$, $-\pi \leqq x \leqq \pi$, is defined as follows:

$$U_1 * U_2 = \int_{-\pi}^{\pi} U_1(x - \xi) U_2(\xi) d\xi.$$

It is evident that the convolution of two even functions is an even function, that the convolution of two odd functions is an even function, and that the convolution of an odd function and an even function is an odd function.

The following theorem is proved by Kniess:[4]

THEOREM. *If $U_1(x)$, $-2\pi \leqq x \leqq 2\pi$, and $U_2(x)$, $-\pi \leqq x \leqq \pi$, are bounded and integrable, if U_1 is odd and periodic with period 2π, and if U_2 is even, then*

Presented to the Society, September 10, 1942; received by the editors March 8, 1944.

[1] The author wishes to thank the referee and Professor R. V. Churchill for many valuable suggestions.

[2] G. Doetsch, *Integration von Differentialgleichungen vermittels der endlichen Fourier Transformation*, Math. Ann. vol. 112 (1935) pp. 52–68.

[3] The lower case letters will be used to signify the transforms of the functions designated by the corresponding capital letters. Instead of using the symbols $u_s(n)$ and $u_c(n)$ we shall use $u(n)$ for both, whenever it is evident which one is meant.

[4] Hans Kniess, *Lösung von Randwertproblemen bei Systemen gewöhnlicher Differentialgleichungen vermittels der endlichen Fourier Transformation*, Math. Zeit. vol. 44 (1938) pp. 266–291.

$$S\{U_1/2 * U_2\} = S\{U_1\}C\{U_2\}.$$

2. **Temperature distribution in a bar.** Consider a cylindrical bar of length π which has a diameter so small that the variation of temperature $U(x, t)$ over every cross section can be neglected. The thermal conductivity of the bar is a function of the time. There is a continuous internal source of heat along the bar and the initial temperature at each point of the bar is given by a prescribed function. The rate of loss of heat through the surface at each point is proportional to the temperature at that point. The temperatures at the ends $x=0$ and $x=\pi$ are determined by prescribed functions of t.

The boundary value problem for the temperature $U(x, t)$ can be written

$$\text{(A)}\qquad \begin{aligned} &L(U) \equiv \partial U/\partial t - C_1(t)\partial^2 U/\partial x^2 + C_2(t)U = P(x, t), && 0 < x < \pi,\ t > 0,\\ &U(+0, t) = C_3(t), \qquad U(\pi - 0, t) = C_4(t), && t > 0,\\ &U(x, +0) = F(x), && 0 < x < \pi. \end{aligned}$$

The prescribed functions P, F, C_1, C_2, C_3, and C_4 are assumed to satisfy the following conditions: P and its first three derivatives with respect to x are continuous functions of x and t for all values of x and t in $0 \leqq x \leqq \pi$, $t \geqq 0$; F and its first three derivatives with respect to x are continuous in x for $0 \leqq x \leqq \pi$; P, F, and their first two derivatives with respect to x vanish at the end points of the interval $(0, \pi)$; C_1 and C_2 are positive and continuous for $t \geqq 0$; C_3' and C_4' are continuous for $t \geqq 0$; $C_3(0) = C_4(0) = 0$.

3. **Resolution of temperature problem.** It is obvious that the solution of our boundary value problem can be written

$$\text{(1)}\qquad U(x, t) = U_1(x, t) + U_2(x, t) + U_3(x, t) + U_4(x, t),$$

where U_1, U_2, U_3, and U_4 are solutions of the problems:

$$\text{(A}_1\text{)}\qquad \begin{aligned} &L(U_1) = P(x, t), && U_1(+0, t) = 0,\\ &U_1(\pi - 0, t) = 0, && U_1(x, +0) = 0; \end{aligned}$$

$$\text{(A}_2\text{)}\qquad \begin{aligned} &L(U_2) = 0, && U_2(+0, t) = C_3(t),\\ &U_2(\pi - 0, t) = 0, && U_2(x, +0) = 0; \end{aligned}$$

$$\text{(A}_3\text{)}\qquad \begin{aligned} &L(U_3) = 0, && U_3(+0, t) = 0,\\ &U_3(\pi - 0, t) = C_4(t), && U_3(x, +0) = 0; \end{aligned}$$

and

$$(A_4)\qquad \begin{aligned} &L(U_4)=0, && U_4(+0,\ t)=0,\\ &U_4(\pi-0,\ t)=0, && U_4(x,\ +0)=F(x). \end{aligned}$$

If in problem (A_2) we replace $C_3(t)$ by $C_4(t)$ and x by $\pi-x$, then $U_2(x,\ t)$ reduces to $U_3(x,\ t)$. For this reason we shall not have to consider problem (A_3) as a problem to be solved.

The finite sine transformation with respect to x applied to the first and fourth equations in problems (A_1), (A_2), and (A_4) transforms each of these boundary value problems into a problem in linear ordinary differential equations of the first order which can be written as follows:

$$(A_1')\qquad \begin{aligned} &d[u_1(n,\ t)]/dt+[n^2C_1(t)+C_2(t)]u_1(n,\ t)=p(n,\ t),\\ &u_1(n,\ +0)=0; \end{aligned}$$

$$(A_2')\qquad \begin{aligned} &d[u_2(n,\ t)]/dt+[n^2C_1(t)+C_2(t)-nC_1(t)C_3(t)]u_2(n,\ t)=0,\\ &u_2(n,\ +0)=0; \end{aligned}$$

and

$$(A_4')\qquad d[u_4(n,\ t)]/dt+[n^2C_1(t)+C_2(t)]u_4(n,\ t)=0,\ u_4(n,\ +0)=f(n).$$

The solutions $u_1(n,\ t)$, $u_2(n,\ t)$, and $u_4(n,\ t)$ of the problems (A_1'), (A_2'), and (A_4'), respectively, are given as follows:

$$u_1(n,\ t)=\int_0^t p(n,\ \tau)\exp\left[-n^2\sigma_1(t,\ \tau)\right]\exp\left[-\sigma_2(t,\ \tau)\right]d\tau,$$

$$u_2(n,\ t)=\int_0^t C_3(\tau)C_1(\tau)\exp\left[-n^2\sigma_1(t,\ \tau)\right]\exp\left[-\sigma_2(t,\ \tau)\right]d\tau,$$

and

$$u_4(n,\ t)=f(n)\exp\left[-n^2s_1(t)\right]\exp\left[-s_2(t)\right],$$

where we have made the definitions: $s_i(t)=\int_0^t C_i(\xi)d\xi$, $\sigma_i(t,\ \tau)=s_i(t)-s_i(\tau)$, $0\leqq\tau\leqq t$.

It will now be shown that each of the problems (A_1), (A_2), and (A_4) can be further resolved into a single simple boundary value problem. That is to say, each of the solutions u_1, u_2, and u_4 can be expressed in terms of $u_0(n,\ t)$, the transform of $U_0(x,\ t)$, which is the solution of the following boundary value problem:

$$(B)\qquad \begin{aligned} &\partial U_0/\partial t=\partial^2U_0/\partial x^2, && 0<x<\pi,\ t>0,\\ &U_0(+0,\ t)=0, \quad U_0(\pi-0,\ t)=0, && t>0,\\ &U_0(x,\ +0)=(\pi-x)/\pi, && 0<x<\pi. \end{aligned}$$

Making a formal application of the finite sine transformation to (B) gives

$$d[u_0(n, t)]/dt + n^2u_0(n, t) = 0, \qquad u_0(n, +0) = 1/n.$$

The solution of this transformed problem is

$$u_0(n, t) = (1/n) \exp(-n^2t).$$

From this solution we obtain the following identities:

$$\exp[-n^2s_1(t)] = nu_0[n, s_1(t)],$$
$$n \exp[-n^2\sigma_1(t, \tau)]C_1(\tau) = \partial u_0[n, \sigma_1(t, \tau)]/\partial\tau,$$
$$n \exp[-n^2s_1(t)]C_1(t) = -\partial u_0[n, s_1(t)]/\partial t.$$

The transforms $u_1(n, t)$, $u_2(n, t)$, and $u_4(n, t)$ can now be expressed as functions of $u_0(n, t)$ as follows:

$$u_1(n, t) = \int_0^t \exp[-\sigma_2(t, \tau)]p(n, \tau)nu_0[n, \sigma_1(t, \tau)]d\tau,$$

$$u_2(n, t) = \int_0^t C_2(\tau) \exp[-\sigma_2(t, \tau)] \frac{\partial}{\partial\tau} u_0[n, \sigma_1(t, \tau)]d\tau,$$

and

$$u_4(n, t) = \exp[-s_2(t)]f(n)nu_0[n, s_1(t)].$$

If the convolution theorem is now applied to $u_1(n, t)$ and $u_4(n, t)$, a formal inverse sine transformation of $u_1(n, t)$, $u_2(n, t)$, and $u_4(n, t)$ can be performed. The result can be written

$$U_1(x, t) = \frac{1}{2} \int_0^t \exp[-\sigma_2(t, \tau)]$$

$$\cdot \int_{-\pi}^{\pi} P(x - \xi, \tau) \frac{\partial}{\partial\xi} U_0[\xi, \sigma_1(t, \tau)]d\xi d\tau, \tag{2}$$

$$U_2(x, t) = \int_0^t C_2(\tau) \exp[-\sigma_2(t, \tau)] \frac{\partial}{\partial\tau} U_0[x, \sigma_1(t, \tau)]d\tau,$$

and

$$U_4(x, t) = \frac{1}{2} \exp[-s_2(t)] \int_{-\pi}^{\pi} F(x - \xi) \frac{\partial}{\partial\xi} U_0[\xi, s_1(t)]d\xi,$$

where P and F are the odd, periodic extensions of period 2π of the original functions P and F, respectively, and $\partial U_0/\partial\xi$ is the even ex-

tension of the original function $\partial U_0/\partial \xi$. The solution U_3 of problem (A_3) can be written

$$U_3(x, t) = \int_0^t C_4(\tau) \exp\left[-\sigma_2(t, \tau)\right] \frac{\partial}{\partial \tau} U_0[\pi - x, \sigma_1(t, \tau)]d\tau.$$

THEOREM. *The general temperature problem* (A) *can be resolved into the solution of the simple temperature problem* (B) *by formulas* (1) *and* (2). *That is,*

$$\begin{aligned} U(x, t) = {} & \frac{1}{2}\int_0^t \exp\left[-\sigma_2(t, \tau)\right] \int_{-\pi}^{\pi} P(x - \xi, \tau) \frac{\partial}{\partial \xi} U_0[\xi, \sigma_1(t, \tau)]d\xi d\tau \\ & + \int_0^t C_4(\tau) \exp\left[-\sigma_2(t, \tau)\right] \frac{\partial}{\partial \tau} U_0[\pi - x, \sigma_1(t, \tau)]d\tau \\ & + \int_0^t C_3(\tau) \exp\left[-\sigma_2(t, \tau)\right] \frac{\partial}{\partial \tau} U_0[x, \sigma_1(t, \tau)]d\tau \\ & + \frac{1}{2} \exp\left[-s_2(t)\right] \int_{-\pi}^{\pi} F(x - \xi) \frac{\partial}{\partial \xi} U_0[\xi, s_1(t)]d\xi. \end{aligned} \tag{3}$$

4. Verification of resolution. Instead of verifying directly that the function given by (3) is a solution of problem (A), we shall take up separately the verifications of the solutions U_1, U_2, and U_4 of the problems (A_1), (A_2), and (A_4), respectively.

Consider problem (A_1) and its solution $U_1(x, t)$ given by (2). By making use of the definitions of extensions, we can write $U_1(x, t)$ as follows:

$$\begin{aligned} U_1(x, t) = {} & \frac{1}{2}\int_0^t \exp(-\sigma_2) \\ & \cdot \int_0^{\pi} \left[P(x + \xi, \tau) + P(x - \xi, \tau)\right] \frac{\partial}{\partial \xi} U_0(\xi, \sigma_1)d\xi d\tau. \end{aligned}$$

Integration by parts with respect to ξ yields

$$U_1(x, t) = \frac{1}{2}\int_0^t \exp(-\sigma_2)H(x, t, \tau)d\tau, \tag{a}$$

where

$$H(x, t, \tau) = \int_0^{\pi} G(x, \xi, \tau)U_0[\xi, \sigma_1(t, \tau)]d\xi, \tag{b}$$

and

$$\text{(c)} \qquad G(x, \xi, \tau) = \left[\frac{\partial}{\partial x} P(x - \xi, \tau) - \frac{\partial}{\partial x} P(x + \xi, \tau)\right].$$

The function $U_0(x, t)$, the solution of problem (B), can be found by an inverse sine transformation of $u_0(n, t)$, or by the usual methods.[5] However, this problem is a special case of a more general problem solved by R. V. Churchill[6] by means of the Laplace transformation. The following properties of $U_0(x, t)$ can be derived from his paper, and are given here without the proofs: $U_0(x, t)$ is of order $O[\exp(\gamma t)]$ uniformly in x, $0 \leqq x \leqq \pi$, for all $t \geqq 0$, for some $\gamma > 0$; $U_0(x, t)$ is continuous in (x, t) when $t \geqq 0$, $0 < x \leqq \pi$; U_0, $\partial U_0/\partial x$, $\partial^2 U_0/\partial x^2$, and $\partial U_0/\partial t$ are continuous in (x, t) for each $t > 0$, $0 \leqq x \leqq \pi$; $|\partial U_0/\partial t|$ is bounded when $t \geqq 0$ and x is in the interval $0 < x_0 \leqq x \leqq \pi$.

That $H(x, t, \tau)$, $\partial H(x, t, \tau)/\partial t$, and $\partial^2 H(x, t, \tau)/\partial x^2$ are continuous functions of (x, t, τ) in R', $0 \leqq x \leqq \pi$, $0 \leqq \tau \leqq t$, can be established as follows:

$F \equiv GU_0$ is a continuous function of $(x, t, \tau, \xi) \equiv (\rho, \xi)$ in the closed region $R'' : R'$, $\delta_1 \leqq \xi \leqq \pi$, $\delta_1 > 0$. Hence $|\Delta F| < \epsilon_1$ for $|\Delta\rho| < \delta'$ in R''. F is uniformly bounded and integrable in the closed region $R''' : R'$, $0 \leqq \xi \leqq \delta_1$. Hence $|F| < M$ in R'''. So that

$$|H(\rho + \Delta\rho) - H(\rho)| \leqq \int_0^{\delta_1} |\Delta F|\, d\xi + \int_{\delta_1}^{\pi} |\Delta F|\, d\xi$$
$$\leqq M\delta_1 + \epsilon_1(\pi - \delta_1) < \epsilon$$

for $|\Delta\rho| < \delta < \delta'$, ρ in R'. This establishes the continuity of $H(x, t, \tau)$.

Consider

$$\frac{\partial}{\partial t} H(x, t, \tau) = \frac{\partial}{\partial t} \int_0^{\pi} G(x, \xi, \tau) U_0[\xi, \sigma_1(t, \tau)]\, d\xi.$$

The derivative of the integral with respect to t yields formally

$$\frac{\partial}{\partial t} H(x, t, \tau) = \int_0^{\pi} G(x, \xi, \tau) \frac{\partial}{\partial t} U_0[\xi, \sigma_1(t, \tau)]\, d\xi$$
$$= C_1(t) \int_0^{\pi} G(x, \xi, \tau) \frac{\partial^2}{\partial \xi^2} U_0(\xi, \sigma_1)\, d\xi,$$

$$\text{(d)} \qquad \frac{\partial}{\partial t} H(x, t, \tau) = C_1(t) \int_0^{\pi} \frac{\partial^2}{\partial \xi^2} G(x, \xi, \tau) U_0(\xi, \sigma_1)\, d\xi.$$

[5] R. V. Churchill, *Fourier series and boundary value problems*, New York, McGraw-Hill Book Company, 1941, p. 102.

[6] R. V. Churchill, *On the problem of temperatures in a non-homogeneous bar with discontinuous initial temperatures*, Amer. J. Math. vol. 61 (1939) pp. 651–664.

This integral is of the type $H(x, t, \tau)$, therefore $\partial H(x, t, \tau)/\partial t$ is a continuous function of (x, t, τ) in R', provided $\partial H(x, t, t)/\partial t$ is defined by (d). The continuity of $\partial^2 H(x, t, \tau)/\partial x^2$ can be established by the same type of argument.

Consider $U_1(x, t)$ as defined by formula (a). Since the integrand is a bounded, integrable function of τ for all $t \geqq 0$, the limit of $U_1(x, t)$ as t approaches zero exists and $U_1(x, +0) = 0$.

Since $G(x, \xi, \tau)$, in (c), vanishes when x approaches both end points of $(0, \pi)$, it follows that $H(0, t, \tau) = H(\pi, t, \tau) = 0$. Hence $U_1(+0, t) = 0$ and $U_1(\pi - 0, t) = 0$.

To show that $U_1(x, t)$ satisfies the differential equation of problem (A_1), we apply Leibnitz' rule to $U_1(x, t)$, in (a), and find that

$$\begin{aligned}\frac{\partial}{\partial t} U_1(x, t) &= \frac{1}{2} \lim_{\tau \to t} \exp(-\sigma_2) H(x, t, \tau) \\ &\quad + \frac{1}{2} \int_0^t \frac{\partial}{\partial t} \{\exp[-\sigma_2(t, \tau)] H(x, t, \tau)\} d\tau \\ &= \frac{1}{2} \int_0^\pi G(x, t, t) U_0(\xi, 0) d\xi - C_2(t) U_1(x, t) \\ &\quad + \frac{1}{2} \int_0^t \exp(-\sigma_2) \frac{\partial}{\partial t} H(x, t, \tau) d\tau.\end{aligned}$$

Since $\partial^2 G/\partial x^2 = \partial^2 G/\partial \xi^2$, we have from (d) that $\partial H/\partial t = C_1 \partial^2 H/\partial x^2$. Hence

$$\begin{aligned}\frac{\partial}{\partial t} U_1(x, t) &= P(x, t) - C_2(t) U_1(x, t) \\ &\quad + C_1(t) \frac{1}{2} \int_0^t \exp(-\sigma_2) \frac{\partial^2}{\partial x^2} H(x, t, \tau) d\tau,\end{aligned}$$

so that

$$\frac{\partial}{\partial t} U_1(x, t) = P(x, t) - C_2(t) U_1(x, t) + C_1(t) \frac{\partial^2}{\partial x^2} U_1(x, t),$$

$$0 < x < \pi, \ t > 0.$$

Consider problem (A_2) and its solution $U_2(x, t)$ given by (2). Since $C_3(0) = 0$, integration by parts of $U_2(x, t)$ yields

$$\begin{aligned}\text{(e)} \quad U_2(x, t) &= C_3(t)((\pi - x)/\pi) \\ &\quad - \int_0^t \exp[-\sigma_2(t, \tau)] U_0[x, \sigma_1(t, \tau)] \{C_3'(\tau) + C_2(\tau) C_3(\tau)\} d\tau,\end{aligned}$$

so that $U_3(x, +0)=0$, $0<x<\pi$.

Now $|U_0|<\epsilon'$ when $0<x<\delta'$, $0\leqq\tau\leqq t-\epsilon''$; $|U_0|<|(\pi-x)/\pi+\epsilon'''|$ when $0<x<\delta''$, $t-\epsilon''\leqq\tau\leqq t$; so that if $E\equiv\exp(-\sigma_2)[C_3'+C_2C_3]$, $|\max E|=M$, then

$$\begin{aligned}|U_3(x,t)-C_3(t)| &\leqq |C_3/\pi|\,x+\left|\int_0^t EU_0\,d\tau\right| \\ &\leqq |C_3|\,x+\left|\int_0^{t-\epsilon''} M\epsilon'\,d\tau\right| \\ &\quad+\left|\int_{t-\epsilon''}^t M|(\pi-x)/\pi+\epsilon'''|\,d\tau\right| \\ &\leqq |C_3|\,x+M(t-\epsilon'')\epsilon'+M|1+\epsilon'''|\,\epsilon''<\epsilon\end{aligned}$$

for $0<x<\delta$, $\delta<\delta'$, δ'', $0<t\leqq T$. Hence $U_3(+0, t)=C_3(t)$, $t>0$. The same type of argument gives $U_3(\pi-0, t)=0$, since $U_0(\pi-0, t)=0$, $t>0$.

In (e), for $t>0$, $0<x<\pi$, first we take the derivative of $U_3(x, t)$ with respect to t, and then we take the second derivative of $U_3(x, t)$ with respect to x. We find that

$$\begin{aligned}\partial U_3(x,t)/\partial t &= C_3'(t)((\pi-x)/\pi)-((\pi-x)/\pi)[C_3'(t)+C_2(t)C_3(t)] \\ &\quad+C_3(t)\int_0^t \exp(-\sigma_2)U_0(x,\sigma_1)[C_3'(\tau)+C_2(\tau)C_3(\tau)]\,d\tau \\ &\quad-C_1(t)\int_0^t \exp(-\sigma_2)(\partial U_0(x,\sigma_1)/\partial\sigma_1)[C_3'(\tau)+C_2(\tau)C_3(\tau)]\,d\tau,\end{aligned}$$

and

$$\begin{aligned}&\partial^2U_3(x,t)/\partial x^2 \\ &\quad=-\int_0^t \exp(-\sigma_2)(\partial^2U_0(x,\sigma_1)/\partial x^2)[C_3'(\tau)+C_2(\tau)C_3(\tau)]\,d\tau.\end{aligned}$$

We form the partial differential expression $L(U_3)$:

$$\begin{aligned}L(U_3) &= \partial U_3/\partial t-C_1(t)\partial^2U_3/\partial x^2+C_2(t)U_3 \\ &= C_1(t)\int_0^t \exp(-\sigma_2)[\partial^2U_0(x,\sigma_1)/\partial x^2 \\ &\qquad-\partial U_0(x,\sigma_1)/\partial\sigma_1][C_3'+C_2C_3]\,d\tau.\end{aligned}$$

Hence $L(U_3)=0$, since $\partial^2U_0(x, \sigma_1)/\partial x^2-\partial U_0(x, \sigma_1)/\partial\sigma_1=0$, $0<x<\pi$, $t>0$.

The verification of $U_4(x, t)$ can be patterned after that used for $U_1(x, t)$ and, therefore, will not be discussed here.

This concludes the verification of the resolution of problem (A). The restrictions placed upon the functions $F(x)$ and $P(x, t)$ are unnecessarily severe. They were made in order to be able to condense the forms of the solutions. The same applies to the condition that $C_3(0) = C_4(0) = 0$. The verification of U_3 and U_4 can be made without them.

5. **Temperature distribution in a cube.** The method of the previous problem can be extended to a general heat flow problem in three dimensions; the same problem (B) is used in the resolution. The verification of the resolution of this problem is similar to that of the first problem and is omitted here.

The interior of a cube[7] of edge length π is filled with a homogeneous, isotropic medium. The temperatures on the faces of the cube are maintained at prescribed values, depending upon the space coordinates of the point and upon the time. There is a continuous source of heat in the medium whose thermal conductivity may be a function of the time. Further, at each point of the medium there is a sink of heat which may be a function of the time and which is proportional to the temperature at that point. The initial temperature distribution is given by a prescribed function. We take a corner of the cube as origin and the three mutually perpendicular edges as coordinate axes to locate a point $P: (x_1, x_2, x_3)$ of the cube.

Let us consider the distribution of temperatures $U(P, t)$ in the interior of such a cube. $U(P, t)$ satisfies the conditions:

$$\frac{\partial}{\partial t} U(P, t) - \sum_{j=1}^{3} C_j(t) \frac{\partial^2}{\partial x_j^2} U(P, t) - C_4(t) U(P, t) = F(P, t), \qquad 0 < x_j < \pi,\ t > 0,$$

$$U(0, x_{2,3}, t) = G_1(x_{2,3}, t),\ U(\pi, x_{2,3}, t) = G_2(x_{2,3}, t),\ 0 \leqq x_{2,3} \leqq \pi,\ t > 0,$$

$$U(0, x_{1,3}, t) = G_3(x_{1,3}, t),\ U(\pi, x_{1,3}, t) = G_4(x_{1,3}, t),\ 0 \leqq x_{1,3} \leqq \pi,\ t > 0,$$

$$U(0, x_{1,2}, t) = G_5(x_{1,2}, t),\ U(\pi, x_{1,2}, t) = G_6(x_{1,2}, t),\ 0 \leqq x_{1,2} \leqq \pi,\ t > 0,$$

$$U(P, 0) = H(P), \qquad 0 < x_j < \pi\ (j = 1, 2, 3).$$

Problem (B), *which is a problem in heat conduction in one dimension, is sufficient to make a formal resolution of this very general problem of heat conduction in three dimensions.*

[7] Compare with problem discussed by Carslaw: H. S. Carslaw, *Introduction to the mathematical theory of the conduction of heat in solids*, London, Macmillan and Company, 1921, p. 108.

If we use the same system of abbreviations as that used before, the solution $U(P, t)$ can be expressed as follows:

$$
\begin{aligned}
U(P,t) = {} & \frac{1}{8}\int_0^t \exp(-\sigma_4)\iiint_{-\pi}^{\pi} F(P_j,\tau)\prod_{j=1}^{4}\frac{\partial}{\partial\xi_j}U_0(\xi_j,\sigma_j)\,d\xi_j d\tau \\
& + \frac{1}{8}\exp(-s_4)\iiint_{-\pi}^{\pi} H(P_j)\prod_{j=1}^{3}\frac{\partial}{\partial\xi_j}U_0(\xi_j,s_j)\,d\xi_j \\
& + \frac{1}{4}\sum_{j=1}^{3}\int_0^t \exp(-\sigma_4)\frac{\partial}{\partial\tau}U_0(x_j,\sigma_j) \\
& \quad \iint_{-\pi}^{\pi} G_{2j-1}(Q_j,\tau)\prod_{k\neq j-1}^{3}\frac{\partial}{\partial\xi_k}U_0(\xi_k,\sigma_k)\,d\xi_k d\tau \\
& + \frac{1}{4}\sum_{j=1}^{3}\int_0^t \exp(-\sigma_4)\frac{\partial}{\partial\tau}U_0(\pi-x_j,\sigma_j) \\
& \quad \cdot \iint_{-\pi}^{\pi} G_{2j}(Q_j,\tau)\prod_{k\neq j-1}^{3}\frac{\partial}{\partial\xi_k}U_0(\xi_k,\sigma_k)\,d\xi_k d\tau,
\end{aligned}
$$

where $(P_j)\equiv(x_1-\xi_1,\ x_2-\xi_2,\ x_3-\xi_3)$ and Q_j is of the form $(x_{j_1}-\xi_{j_1},\ x_{j_2}-\xi_{j_2})$; $j_1, j_2\neq j=1, 2, 3$.

RENSSELAER POLYTECHNIC INSTITUTE

A SHORT PROOF OF THE COMPLETENESS OF THE LAGUERRE FUNCTIONS

LYNN H. LOOMIS

The Laguerre functions $\phi_n(x)$, $n=0, 1, \cdots$, can be defined[1] as the coefficients of t^n in the power series expansion about $t=0$ of the generating function

$$F(x, t) = \exp\left(-\frac{x}{2}\left(\frac{1+t}{1-t}\right)\right)\Big/(1-t).$$

THEOREM 1. *The functions $\phi_n(x)$ form an orthonormal set over $(0, \infty)$.*

The following proof is essentially well known.[2] From the definition of $\phi_n(x)$ we see that $\int_0^\infty \phi_m(x)\phi_n(x)dx$ is the coefficient of $t^m s^n$ in the power series expansion about $(t, s)=(0, 0)$ of

$$\int_0^\infty F(x, t)F(x, s)dx = \int_0^\infty \left(\exp\left(-x\frac{1-ts}{(1-t)(1-s)}\right)\right)\Big/(1-t)(1-s))dx$$

$$= \frac{1}{1-ts} = \sum_{p=0}^{\infty}(ts)^p.$$

Hence $\int_0^\infty \phi_m(x)\phi_n(x)dx$ is 1 or 0 according as $m=n$ or $m\neq n$, as was to be proved.

The interchange of the order of integration and summation can be justified by first making the interchange for the finite integral and then letting the upper limit approach infinity.

THEOREM 2. *The functions $\phi_n(x)$ are complete (closed) in $L^2(0, \infty)$.*

We use a special case of the well known theorem[3] that if $f(z) = a_0+a_1z+a_2z^2+\cdots$ is analytic and bounded in $|z|<1$ with boundary function $f(e^{i\theta})$, then

$$\int_0^{2\pi} |f(e^{i\theta})|^2 d\theta = 2\pi\sum_{n=0}^{\infty}|a_n|^2. \tag{1}$$

Since step functions are dense in $L^2(0, \infty)$, it is sufficient, in order to prove the completeness of the $\phi_n(x)$, to prove that the closed linear

Received by the editors October 8, 1943.

[1] See Kaczmarz and Steinhaus, *Theorie der Orthogonalreihen*, p. 140, line 1, (55) and [482].

[2] See, for example, G. Szegö, *Orthogonal polynomials*, Amer. Math. Soc. Colloquium Publications, vol. 23, p. 69.

[3] See Bieberbach, *Lehrbuch der Funktionentheorie*, vol. 1, p. 145.

space which they determine includes the step functions, hence that it includes the characteristic function $g_b(x)$ of the interval $0 \leqq x \leqq b$. Thus if a_n is the nth Laguerre coefficient of $g_b(x)$,

$$a_n = \int_0^\infty g_b(x)\phi_n(x)dx = \int_0^b \phi_n(x)dx, \tag{2}$$

we have to prove that

$$\int_0^\infty \left(g_b(x) - \sum_{n=0}^{N} a_n\phi_n(x)\right)^2 dx \to 0$$

as $N \to \infty$, that is, that

$$b = \sum_{n=0}^{\infty} |a_n|^2. \tag{3}$$

By (2) the Laguerre coefficients a_n of $g_b(x)$ are the coefficients of t^n in the expansion of

$$f(t) = \int_0^b \left(\exp\left(-\frac{x}{2}\left(\frac{1+t}{1-t}\right)\right)\Big/(1-t)\right)dx$$

$$= -\frac{2}{1+t}\left[\exp\left(-\frac{b}{2}\left(\frac{1+t}{1-t}\right)\right) - 1\right].$$

This function is analytic in $|t| \leqq 1$ except at $t=1$, and is bounded in $|t| < 1$. Putting $t = e^{i\theta}$ we have

$$f(e^{i\theta}) = -(1 - i\tan(\theta/2))\left(\exp\left(-\frac{b}{2}i\cot\frac{\theta}{2}\right) - 1\right),$$

and the substitutions $y = \tan(\theta/2)$, $x = b/4y$ give

$$\int_0^{2\pi} |f(e^{i\theta})|^2 d\theta = 8\int_{-\infty}^{\infty} \sin^2(b/4y)dy = 2b\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}dx = 2\pi b. \tag{4}$$

The desired result (3) follows from (1) and (4).

HARVARD UNIVERSITY

ON THE GROWTH OF SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS

J. ERNEST WILKINS, JR.

1. Introduction. In a recent paper by Boas, Boas and Levinson [1][1] two sets of sufficient conditions were given for the existence of $\lim_{x\to\infty} y'(x)$ when $y(x)$ satisfies the differential equation

$$y'' + A(x)y = B(x). \tag{1:1}$$

We propose in this paper to use their methods and to generalize their results to the nth order linear differential equation

$$y^{(n)} + \sum_{i=1}^{n} A_i(x)y^{(n-i)} = B(x), \tag{1:2}$$

and to obtain sufficient conditions for

$$\lim_{x\to\infty} y^{(n-1)}(x) \tag{1:3}$$

to exist. In case $n=2$, $A_1(x)=0$ and $A_2(x)=A(x)$, these conditions reduce to those in [1].

2. Statements of the theorems. In §4 we shall prove the following theorem.

THEOREM I. *If $A_i(x)$ $(i=1, \cdots, n)$ and $B(x)$ are continuous on $0\leqq x<\infty$, and if the integrals*

$$\int_0^\infty x^{i-1}\left|A_i(x)\right|dx \qquad (i=1, \cdots, n), \tag{2:1}$$

$$\int_0^\infty B(x)dx \tag{2:2}$$

exist, then the limit (1:3) *exists for any solution $y(x)$ of* (1:2).

We now write each function $A_i(x)$ as the difference of two non-negative functions, $A_i(x)=A_i'(x)-A_i''(x)$, where $A_i'=(|A_i|+A_i)/2$, $A_i''=(|A_i|-A_i)/2$. Then in §5, $\cdots$, §8 we shall prove the following theorem.

THEOREM II. *If $A_i(x)$ $(i=1, \cdots, n)$ and $B(x)$ are continuous on $0\leqq x<\infty$, if the integrals*

Received by the editors October 30, 1943.

[1] Numbers in brackets refer to the Bibliography at the end of the paper.

(2:3) $$\int_0^\infty x^{i-1}A_i''(x)dx \qquad (i = 1, \cdots, n),$$

(2:4) $$\int_0^\infty B(x)dx$$

exist, and if we have

(2:5) $$\limsup_{x=\infty} x^{i-k-1}\int_x^\infty t^k A_i'(t)dt < 2(i-k-1)!k!/n(n-1)$$

whenever $i=2$, $k=0$, *or* $i=2j-1$, $2j$, $k=i-j, \cdots, i-2$ $(j=2, \cdots, [(n+1)/2])$, *where we agree that* $A_{n+1}'(t)=0$ *if* n *is odd, then the limit* (1:3) *exists for any solution* $y(x)$ *of* (1:2). *If in addition we have*

(2:6) $$\int_0^\infty \sum_{i=1}^n x^{i-1}A_i'(x)dx = \infty,$$

then $\lim_{x=\infty} y^{(n-1)}(x)=0$.

3. **Some auxiliary lemmas.** In this section we state four lemmas needed in proving the main theorems. The first of these is found in [1].

LEMMA 1. *If* $f(x)$ *is continuous on* $0\leqq x<\infty$, *if* $M(x)$ *denotes the maximum of* $|f(t)|$ *on* $0\leqq t\leqq x$, *and if, for some positive numbers* α *and* x_0, $|f(x)|\leqq\alpha+M(x)/2$ $(x\geqq x_0)$, *then* $f(x)$ *is bounded on* $0\leqq x<\infty$.

LEMMA 2. *If* (2:5) *holds under the restrictions on* i *and* k *stated in Theorem* II, *then* (2:5) *also holds for* $i=2, \cdots, n$, $k=0, \cdots, i-2$.

This is manifestly true for $i=2$. If $i>2$ and if i is odd, then $i=2j-1$, $j\geqq 2$ and (2:5) holds if $k\geqq i-j=j-1$. Suppose now that $0\leqq k<j-1$. Then $i-k-1=2j-2-k>j-1$, and

$$x^{i-k-1}\int_x^\infty t^k A_i'(t)dt \leqq x^{j-1}\int_x^\infty t^{j-1}A_i'(t)dt,$$

$$\limsup_{x=\infty} x^{i-k-1}\int_x^\infty t^k A_i'(t)dt < 2(j-1)!^2/n(n-1) < 2(2j-2-k)!k!/n(n-1).$$

The reasoning when i is even is quite similar.

LEMMA 3. *If* $f(x)$ *is continuous and bounded on* $0\leqq x<\infty$, *and if the functions* $A_i(x)$ *satisfy the hypotheses of Theorem* I, *then all integrals of the form*

$$\int_0^\infty f(x)x^p A_i(x)dx, \qquad \int_0^\infty x^p \left| A_i(x) \right| dx \qquad (p = 0, \cdots, i-1)$$

exist. Under the hypotheses of Theorem II, *the same conclusion is valid provided that* $i=2, \cdots, n$, $p=0, \cdots, i-2$, *or that* $A_i(x)$ *be replaced by* $A_i'(x)$.

The proof of the first sentence of the lemma is immediate, and the second sentence follows similarly as soon as we refer to the preceding lemma.

LEMMA 4. *If* $y(x)$ *is of class* $C^{(m+q)}$ *on* $0 \leqq x < \infty$, *where* m *and* q *are non-negative integers, then*

$$q! \limsup_{x=\infty} x^{-q} \left| y^{(m)}(x) \right| \leqq \limsup_{x=\infty} \left| y^{(m+q)}(x) \right|. \tag{3:1}$$

To prove Lemma 4 we use Taylor's Theorem in the form

$$\begin{aligned} y^{(m)}(x) = &\sum_{k=0}^{q-1} (x - x_0)^k y^{(m+k)}(x_0)/k! \\ &+ \int_{x_0}^{x} \int_{x_0}^{t_{q-1}} \cdots \int_{x_0}^{t_1} y^{(m+q)}(t_0)dt_0dt_1 \cdots dt_{q-1}. \end{aligned} \tag{3:2}$$

Let $M = \limsup_{x=\infty} \left| y^{(m+q)}(x) \right|$ and pick $e>0$. Then take x_0 so large that

$$\left| y^{(m+q)}(x) \right| < M + e \qquad (x \geqq x_0).$$

It follows from (3:2) that if $x \geqq x_0$

$$\left| y^{(m)}(x) \right| \leqq \sum_{k=0}^{q-1} x^k \left| y^{(m+k)}(x_0) \right| / k! + (M + e)x^q/q!,$$

$$q! \limsup_{x=\infty} x^{-q} \left| y^{(m)}(x) \right| \leqq M + e.$$

Since e is arbitrary, the statement of the lemma follows at once.

4. **Proof of Theorem** I. By virtue of (2:1) we can pick x_0 such that

$$\int_{x_0}^{\infty} \sum_{i=1}^{n} x^{i-1} \left| A_i(x) \right| / (i-1)!dx < 1/2. \tag{4:1}$$

If in (1:2) we substitute for $y^{(n-i)}$ $(i=2, \cdots, n)$ the values obtained from (3:2) by replacing m by $n-i$ and q by $i-1$, and solve the resulting equation for $y^{(n)}$, we get an equation which upon integration between the limits x_0 and x gives

$$y^{(n-1)}(x) = y^{(n-1)}(x_0) - \sum_{i=2}^{n}\sum_{k=0}^{i-2}\int_{x_0}^{x} A_i(t)\,\frac{(t-x_0)^k}{k!}\,y^{(n-i+k)}(x_0)\,dt$$

$$- \sum_{i=1}^{n}\int_{x_0}^{x}\int_{x_0}^{t_{i-1}}\cdots\int_{x_0}^{t_1} A_i(t_{i-1})\,y^{(n-1)}(t_0)\,dt_0\cdots dt_{i-1} \tag{4:2}$$

$$+ \int_{x_0}^{x} B(t)\,dt.$$

Define the quantities B and α and the function $M(x)$ by the equations

$$B = \max_{x_0\le z\le\infty}\left|\int_{x_0}^{z} B(t)\,dt\right|,$$

$$\alpha = \left|\,y^{(n-1)}(x_0)\,\right| \tag{4:3}$$

$$+ \sum_{i=2}^{n}\int_{x_0}^{\infty}\left|\,A_i(t)\,\right|\sum_{k=0}^{i-2}\frac{t^k}{k!}\left|\,y^{(n-i+k)}(x_0)\,\right|dt + B,$$

$$M(x) = \max_{0\le t\le x}\left|\,y^{(n-1)}(t)\,\right|.$$

B exists by virtue of (2:2) and α exists by virtue of Lemma 3. From (4:2) and (4:1) we now get

$$\left|\,y^{(n-1)}(x)\,\right| \le \alpha + M(x)/2 \qquad (x \ge x_0). \tag{4:4}$$

It follows from Lemma 1 that $y^{(n-1)}(x)$ is bounded on $0\le x<\infty$. In this event we use Lemma 3 to see that the integrals involving $A_i(t)$ on the right side of (4:2) approach limits as $x\to\infty$. By (2:2) the integral of $B(x)$ approaches a limit. Therefore, $y^{(n-1)}(x)$ has a limit, proving Theorem I.

5. **Proof of Theorem II when $y^{(n-1)}(x)$ does not change sign for large values of x.** Then we may assume without loss of generality that x_0 is so large that $y^{(n-1)}(x)\ge 0$ for $x\ge x_0$ and that

$$\int_{x_0}^{\infty}\sum_{i=1}^{n} x^{i-1}\left|A_i''(x)/(i-1)!\right|dx < 1/2. \tag{5:1}$$

Since $y^{(n-1)}(t)\ge 0$ on $t\ge x_0$ and $-A_i(t) = A_i'(t) - A_i'(t) \le A_i''(t)$, we then have from (4:2) and (5:1) that (4:4) holds, the integrals in α existing by virtue of the second part of Lemma 3. It follows from Lemma 1 that $y^{(n-1)}(x)$ is bounded. Set $A_i = A_i' - A_i''$ in (4:2). Since $y^{(n-1)}(x)$ is bounded we see from Lemma 2 that all of the terms in (4:2) on the right side approach limits as $x\to\infty$ with the possible exception of

$$(5:2) \qquad -\sum_{i=1}^{n}\int_{x_0}^{x}\int_{x_0}^{t_{i-1}}\cdots\int_{x_0}^{t_1}A_i'(t_{i-1})y^{(n-1)}(t_0)dt_0\cdots dt_{i-1}.$$

Since $A_i' \geqq 0$, $y^{(n-1)}(t_0) \geqq 0$ for $t_0 \geqq x_0$, this term is a nonincreasing function of x which is bounded below since all the other terms in (4:2) are bounded. Hence it also approaches a limit. Therefore, the limit (1:3) exists.

6. **Proof of Theorem II when $y^{(n-1)}(x)$ changes sign infinitely many times.** Suppose first that $y^{(n-1)}(x)$ is bounded but that the limit (1:3) does not exist. Then we may assume without loss of generality that

$$\limsup \left| y^{(n-1)}(x) \right| = \limsup y^{(n-1)}(x) = M > 0.$$

Let x_m be a monotone sequence of points such that $x_m \to \infty$, $y^{(n-1)}(x_m) > 0$, $y^{(n-1)}(x_m) \to M$. Let a_m be the first point to the left of x_m such that $y^{(n-1)}(a_m) = 0$. We can suppose that a_1 is so large that for some $c < 1$ and for $i = 2, \cdots, n$, $k = 0, \cdots, i-2$ we have

$$(6:1) \qquad x^{i-1-k}\int_x^{\infty} t^k A_i'(t)dt < 2c(i-1-k)!k!/n(n-1) \quad (x \geqq a_1).$$

By (4:2) with x_0 replaced by a_m and x replaced by x_m we have, if we observe that $y^{(n-1)}(t) \geqq 0$ on $a_m \leqq t \leqq x_m$,

$$\begin{aligned} y^{(n-1)}(x_m) \leqq & \sum_{i=1}^{n}\int_{a_m}^{x_m}\int_{a_m}^{t_{i-1}}\cdots\int_{a_m}^{t_1}A_i''(t_{i-1})y^{(n-1)}(t_0)dt_0\cdots dt_{i-1} \\ & + \sum_{i=2}^{n}\sum_{k=0}^{i-2}\int_{a_m}^{x_m}\frac{t^k}{k!}A_i''(t)\left| y^{(n-i+k)}(a_m)\right| dt \\ & + \sum_{i=2}^{n}\sum_{k=0}^{i-2}\int_{a_m}^{x_m}\frac{t^k}{k!}A_i''(t)\left| y^{(n-i+k)}(a_m)\right| dt \\ & + \left|\int_{a_m}^{x_m}B(t)dt\right|. \end{aligned} \qquad (6:2)$$

Since $y^{(n-1)}(t)$ is bounded, we see from Lemma 3 that the first sum on the right of (6:2) approaches zero as $m \to \infty$. By virtue of (2:4) so does the last term in (6:2). If we use Lemma 4 we discover that the upper limit of the third sum in (6:2) can not exceed

$$\begin{aligned} \limsup_{m=\infty} & \sum_{i=2}^{n}\sum_{k=0}^{i-2}\frac{\left| y^{(n-i+k)}(a_m)\right|}{k!a_m^{i-1-k}}\int_{a_m}^{x_m}t^{i-1}A_i''(t)dt \\ & \leqq \limsup_{x=\infty} \sum_{i=2}^{n}\sum_{k=0}^{i-2}\frac{\left| y^{(n-1)}(x)\right|}{k!(i-1-k)!}\int_x^{\infty}t^{i-1}A_i''(t)dt = 0. \end{aligned}$$

Finally we use (6:1) and Lemma 4 to see that the upper limit of the second sum in (6:2) cannot exceed

$$\limsup_{m\to\infty} \sum_{i=2}^{n}\sum_{k=0}^{i-2}\int_{a_m}^{\infty} \frac{t^k}{k!} A_i'(t)\left|\, y^{(n-i+k)}(a_m)\,\right| dt$$

$$\leqq \limsup_{x\to\infty} \sum_{i=2}^{n}\sum_{k=0}^{i-2} \frac{2c(i-1-k)!\left|\, y^{(n-i+k)}(x)\,\right|}{n(n-1)x^{i-1-k}}$$

$$\leqq \limsup_{x\to\infty} \sum_{i=2}^{n}\sum_{k=0}^{i-2} 2c\left|\, y^{(n-1)}(x)\,\right|/n(n-1) = cM < M.$$

Referring to (6:2) we see that we have reached a contradiction of our choice of the points x_m.

7. **Proof that $y^{(n-1)}(x)$ must be bounded.** To complete the proof of the first part of Theorem II, it is sufficient to prove that $y^{(n-1)}(x)$ must be bounded under the hypotheses (2:3), (2:4), (2:5) and the assumption that $y^{(n-1)}(x)$ changes sign infinitely many times. Suppose on the contrary that $y^{(n-1)}$ is unbounded. Then we can pick a sequence $x_m\to\infty$ such that

$$\left|\, y^{(n-1)}(x_m)\,\right| \geqq \left|\, y^{(n-1)}(x)\,\right| \qquad (x \leqq x_m), \tag{7:1}$$

$y^{(n-1)}(x_m)$ has the same sign, which we may suppose to be positive, and $y^{(n-1)}(x_m)\to\infty$. Let a_m be defined for x_m as in §6, and suppose that a_1 is so large that (6:1) holds. Using (7:1) and Taylor's Theorem (3:2) with m replaced by $n-i+k$, q replaced by $i-1-k$, x_0 replaced by 0, and x replaced by a_m, we find that

$$\left|\, y^{(n-i+k)}(a_m)\,\right| \leqq y^{(n-1)}(x_m)\ \frac{a_m^{i-k-1}}{(i-k-1)!} + \sum_{h=0}^{i-k-2}\left|\, y^{(n-i+k+h)}(0)\,\right| \frac{a_m^h}{h!}.$$

It now follows from (6:2) and (6:1) that

$$y^{(n-1)}(x_m) \leqq y^{(n-1)}(x_m)\left\{ \sum_{i=1}^{n}\int_{a_m}^{\infty} \frac{t^{i-1}A_i''(t)}{(i-1)!}\,dt + c \right.$$

$$\left. + \sum_{i=2}^{n}\sum_{k=0}^{i-2}\int_{a_m}^{\infty} \frac{t^{i-1}A_i''(t)}{k!(i-k-1)!}\,dt \right\} + \left|\int_{a_m}^{x_m} B(t)dt\right|$$

$$+ \sum_{i=2}^{n}\sum_{k=0}^{i-2}\sum_{h=0}^{i-k-2} \frac{\left|\, y^{(n-i+k+h)}(0)\,\right|}{k!h!}\int_{a_m}^{\infty} t^{k+h}A_i(t)dt.$$

Since all of the integrals on the right of this last inequality approach zero as $m\to\infty$ and $0<c<1$, we reach a contradiction.

8. Proof of the second part of Theorem II. Suppose on the contrary that $y^{(n-1)}(x)$ does not approach zero. Without loss of generality we may assume that

$$\lim_{x=\infty} y^{(n-1)}(x) = 2a > 0.$$

Then there exists an x_0 such that

$$3a > y^{(n-1)}(x) > a \qquad (x \geqq x_0). \tag{8:1}$$

Now set $A_i = A_i' - A_i''$ in (4:2) and let $x \to \infty$. Then all of the terms on the right approach limits with the possible exception of the term (5:2). Since $y^{(n-1)}(x)$ approaches a limit, so must (5:2). But by (8:1) we have

$$\int_{x_0}^{t_{i-1}} \cdots \int_{x_0}^{t_1} y^{(n-1)}(t_0)dt_0 \cdots dt_{i-2} > a(t_{i-1} - x_0)^{i-1}/(i-1)!.$$

Consequently, the term (5:2) is greater than

$$a \int_{x_0}^{x} \sum_{i=1}^{n} (t - x_0)^{i-1} A_i'(t)/(i-1)!dt.$$

By (2:6) and Lemma 3 this last integral becomes infinite as $x \to \infty$, so that (5:2) cannot approach a limit. This contradiction completes the proof of Theorem II.

Added in proof. Since the submission of this paper to the editors, it has come to the author's attention that Theorem I was proved by Otto Haupt, *Über das asymptotische Nerhalten der Lösungen gewisser linearer gewöhnlicher Differentialgleichungen*, Math. Zeit. vol. 48 (1942) pp. 282–292. Our proof, based on Lemma 1, seems distinctly simpler and certainly more elementary than that of Haupt. To the best of our present knowledge, Theorem II is new.

Bibliography

1. Boas, Boas and Levinson, *The growth of solutions of a differential equation*, Duke Math. J. vol. 9 (1942) pp. 847–853.

Tuskegee Institute

LIMIT POINTS OF SUBSEQUENCES

R. CREIGHTON BUCK[1]

1. Introduction. In a previous paper [2],[2] it was shown that for simple sequences of real numbers, divergence of a sequence implies divergence of almost every subsequence. The proof given there required in an essential way that the space be metric. The purpose of this note is to show that the above result holds for multiple sequences in an L^* space. If the space is compact separable metric, even more is true: *the set of limit points of almost every subsequence coincides with the set of limit points of the original sequence.*

2. Notation. We denote by $x=x[n]=x[n_1, n_2, \cdots, n_r]$, where $n_k=1, 2, \cdots$, an arbitrary r-tuple sequence with terms in a space M. Likewise, $x'=x_\lambda[n]=x[\lambda_1(n_1), \lambda_2(n_2), \cdots, \lambda_r(n_r)]$ will denote an arbitary subsequence of x; here, for any fixed k, $\lambda_k(1), \lambda_k(2), \cdots$ form an increasing sequence of integers. This is the natural generalization of subsequences x_{λ_n} of the simple sequence x_n.

Let $\mathfrak{S}$ be the set of all sequences $s=(s_n)$ composed of 0's and 1's, containing infinitely many 1's. This represents in the usual manner the class of subsequences of a simple sequence [1, p. 788] that is $s_n=1$ if x_n is chosen, and 0 if x_n is omitted. A product measure can be defined in $\mathfrak{S}$ [4, p. 420; 5, p. 144].[3]
Then

$$\mathfrak{S}=\mathfrak{S}\times\mathfrak{S}\times\cdots\times\mathfrak{S} \qquad (r \text{ factors})$$

is the class of all subsequences of an r-tuple sequence. Measure is defined in $\mathfrak{S}$ as the product measure over $\mathfrak{S}$.

We assume that some definition of convergence is given for sequences in M. Limit points are then defined as

$$Px = p[\lim x' = p \text{ for some subsequence } x' \text{ of } x].$$

We recall the defining conditions of the Fréchet limit spaces [3, p. 77]:

(L) $\lim x=p$ *implies* $\lim x'=p$ *for every subsequence* x' *of* x.

(L^*) $\lim x=p$ *if and only if* $p\in Px'$ *for every subsequence* x' *of* x.

L^* is the "star-convergence" of Urysohn.

Presented to the Society, October 30, 1943, under the title *Multiple sequences*; received by the editors December 3, 1943.

[1] Society of Fellows, Harvard University.

[2] Numbers in brackets refer to the references listed at the end of the paper.

[3] $\mathfrak{S}\subseteq\mathfrak{S}^*=S\times S\times S\times S\times\cdots$, where $S=\{0, 1\}$. Both $\mathfrak{S}$ and $\mathfrak{S}^*$ have measure 1. $\mathfrak{S}$ can be mapped, with measure preserved, on the real numbers $0<t\leq 1$.

3. **Limit points.** The following lemma is essential.

LEMMA. *Almost every $s \in \mathfrak{S}$ coincides with any fixed $t \in \mathfrak{S}$ in an infinite number of* 1 *coordinates.*

Set $s=(s_\alpha)$ and $t=(t_\alpha)$; t_α is 1 for an infinite set of α's. Let

$$F_n = \mathfrak{s}[s_\alpha t_\alpha = 0 \text{ for all } \alpha > n],$$
$$C_n = \mathfrak{s}[\sum s_\alpha t_\alpha = n].$$

Then F_n is an elementary subset of $\mathfrak{S}$, and furthermore $m(F_n)=0$; since $C_n \subseteq \bigcup_{k \geq n} F_k$, $m(C_n)=0$. If $C=\bigcup C_n$, then $m(C)=0$ so that $m(C')=1$ where

$$C' = \mathfrak{s}[\sum s_\alpha t_\alpha = \infty].$$

THEOREM 1. *If M is an L space and $p \in Px$, then $p \in Px'$ for almost every subsequence x' of x.*

If $p \in Px$, then for some subsequence x'' of x, $\lim x''=p$. Let τ be the point of $\mathfrak{S}$ corresponding to x'', and $\tau=(t^1, t^2, t^3, \cdots, t^r)$. By the lemma, for each k there is a set Σ_k having measure 1, composed of points $s^k=(s^k_\alpha)$ of $\mathfrak{S}$ whose coordinates coincide with those of $t^k=(t^k_\alpha)$ at an infinite number of 1's. Then

$$\Sigma = \Sigma_1 \times \Sigma_2 \times \cdots \times \Sigma_r$$

is a subset of $\mathfrak{S}$ composed of points $\sigma=(s^1, s^2, s^3, \cdots, s^r)$ each of which coincides with τ in an infinite number of components. Σ then corresponds to a set of subsequences x' of x each of which has a subsequence in common with x''. By (L), this common subsequence converges to p, and $p \in Px'$. This is true for almost every subsequence x', for $m(\Sigma)=1$.

THEOREM 2. *If M is an L^* space with unique limits, then if x is divergent, so is almost every subsequence x' of x.*[4]

Either Px is void, or $Px=\{p\}$, or Px contains at least two distinct points. If Px is void, then *every* subsequence of x is divergent. If Px contains distinct points p and q, then by the previous theorem, these are limit points of almost every subsequence, and since limits are unique, these are in turn divergent. Finally, if x is divergent and $Px=\{p\}$, there must be a subsequence x'' of x for which Px'' is void. As in the proof of Theorem 1, almost every subsequence x' of x has a subsequence in common with x'', and must therefore be divergent.

[4] The statements of this and the following theorem were altered at the suggestion of Dr. M. M. Day.

We now suppose that closure of subsets of M is defined.

THEOREM 3. *If M is an L space and Px is separable, then $\overline{Px'} = \overline{Px}$ for almost every subsequence x' of x.*

Choose a countable dense set $E \subset Px$. Let A_p be the class of subsequences of x having p as a limit point. Theorem 1 states that if $p \in Px$, $m(A_p) = 1$. Set $A = \bigcap_{p \in E} A_p$; since E is countable, $m(A) = 1$. Thus, for almost every subsequence x' of x, $E \subset Px' \subset Px$. Closing these sets, we have the theorem.

If M is a neighborhood space, and we define convergence in the usual way, M becomes an L^* space. Thus, Theorem 1 holds for multiple sequences in any topological space. If, further, M obeys the Hausdorff separation axiom, limits are unique and Theorem 2 holds. If M is a topological space with second countability, then Px is separable, and Theorem 3 holds.

COROLLARY 1. *If M is a topological space with second countability, such that the interior of any neighborhood of a point contains a neighborhood of that point, then for simple sequences x, $Px = Px'$ for almost all subsequences x' of x.*

The additional assumption implies that for simple sequences, the set of limit points is closed. (It might be noted that, as subsequences and limit points are here defined, the set of limit points of a multiple sequence need not be closed!)

COROLLARY 2. *If, in addition to the assumptions above, M is also a group, then almost every bracketing of the series $\sum a_n$ leaves the set of limit points unchanged.*

As in [2], bracketing a series is equivalent to selecting a subsequence.

REFERENCES

1. Birnbaum and Zuckerman, *On the properties of a collective*, Amer. J. Math. vol. 62 (1940) pp. 787–791.

2. R. C. Buck and Harry Pollard, *Convergence and summability properties of subsequences*, Bull. Amer. Math. Soc. vol. 49 (1943) pp. 924–931.

3. Kuratowski, *Topologie*. I, Warsaw, 1933.

4. van Kampen, *Infinite product measures and infinite convolutions*, Amer. J. Math. vol. 62 (1940) pp. 417–448.

5. C. Visser, *Law of nought or one*, Studia Math. vol. 7 (1938) pp. 143–159.

HARVARD UNIVERSITY

CLUSTER POINTS OF SUBSEQUENCES[1]

M. M. DAY

In the preceding paper [1][2] Buck defines a class of "subsequences" of a multiple sequence and shows that "almost all" of such subsequences have certain properties. This note is essentially based on a different choice of the definition of "subsequences"; that is, this paper and [1] are generalizations in different directions of a preceding paper by Buck and Pollard (reference 2 of [1]). In this discussion countability is the important property of the index systems such as the integers underlying the simple sequences or the n-tuples of integers underlying the multiple sequences. Countability is a slightly stronger condition than is necessary since the results will be shown to hold as well for functions of n variables as for multiple sequences; some other special cases are mentioned at the end of this paper. Also I modify Buck's approach by considering cluster points in neighborhood spaces rather than limit points in convergence spaces [3]. It may be mentioned that even for multiple sequences Theorems 1 and 2 of these papers are independent since Buck's set of "subsequences" is a set of measure zero in the set of "subsequences" considered here; my Theorem 3 contains the corresponding theorem of [1] as a special case. Lemma 1 and its corollary, Lemma 3, are the fundamental results on which the theorems rest; Lemma 3 is the generalization appropriate to this paper of the lemma in §3 of [1].

1. **Preliminaries.** If R is any set, a product measure can be defined in the set of characteristic functions of subsets of R [1, footnote 2] and this in turn induces a measure $|\cdots|$ for subsets of the set $\mathcal{E}$ of all subsets E of R; this measure is non-negative, completely additive, and (if R is infinite) takes all values between 0 and 1 inclusive; its other principal characteristic is that if $r_1, \cdots, r_k \in R$, then[3] $\{E \mid \text{no } r_i \in E\}$ is of measure 2^{-k}; hence if E_0 is an infinite subset of R and $A = \{E \mid E \cap E_0 \text{ is empty}\}$, $|A| = 0$.

An *index system* $\mathfrak{R} = (R, \geqq)$ is a set R and a binary relation $\geqq$ such that $\geqq$ is transitive and every element r_0 has a successor $r_1 > r_0$ such that $r_0 \not\geqq r_1$. (In the language of [4] $\mathfrak{R}$ is oriented and has no terminal

Presented to the Society, April 29, 1944; received by the editors December 13, 1943.

[1] Considerations suggested by the preceding paper of R. C. Buck.

[2] Numbers in brackets refer to the Bibliography at the end of the paper.

[3] The usual notation of $\bigcup$ and $\bigcap$ will be used for union and intersection of sets; $\{p \mid P\}$ will mean the set of all p having the property P.

elements.) A set E in $\mathfrak{R}$ is called *cofinal* in $\mathfrak{R}$ if for every r in R there exists $r' \geqq r$ with r' in E. Let $E^* = \{r \mid r \geqq \text{some } r' \text{ in } E\}$.

Note that if $\mathfrak{R}$ is the system of integers ordered by magnitude, then the cofinal subsets of $\mathfrak{R}$ are the infinite subsets; for g defined on a general index system $\mathfrak{R}$ it is clear that reducing the domain of definition of g to a cofinal subset E of $\mathfrak{R}$ is a generalization of the process of selecting a subsequence in case $\mathfrak{R}$ is the system of integers.

A subsystem $\mathfrak{R}' = (R', \geqq)$ of $\mathfrak{R} = (R, \geqq)$ is a subset R' of R with the order relation between points of R' defined by that in R; if R' is cofinal in the index system $\mathfrak{R}$, then $(R', \geqq)$ is also an index system. Cofinality is transitive in a transitive system; that is, if R' is cofinal in $(R, \geqq)$ and R'' is cofinal in $(R', \geqq)$, then R'' is cofinal in $(R, \geqq)$.

We may note that if $\mathfrak{R}$ is the set of n-tuples of integers (the case studied in [1]), where $(i_1, \cdots, i_n) \geqq (j_1, \cdots, j_n)$ if and only if $i_k \geqq j_k$ for every $k \leqq n$, then the product subsets defined by Buck are cofinal in $\mathfrak{R}$ but are very sparsely distributed in the set of all cofinal subsets of $\mathfrak{R}$; to be precise, such sets form a set of measure 0 if $n \geqq 2$. A product set in the set of n-tuples of integers, $R = I \times I \times \cdots \times I$, is a set of the form $E_1 \times E_2 \times \cdots \times E_n$, $E_i \subset I$; these product sets define the class of "subsequences" used by Buck. If E_k is the set of elements of R with all coordinates not greater than k and if A_k is the class of all subsets E of R such that $E \cap E_k$ is a product set in E_k, then $A = \bigcap_k A_k$. It is easily seen that if $E' \subset E_k$, $\{E \mid E \cap E_k = E'\}$ is of measure 2^{-k^n}; since there are $(2^k - 1)^n + 1 < 2^{nk}$ product sets in E_k, it follows that $|A_k| < 2^{nk - k^n}$; if $n \geqq 2$, this tends to zero as k increases so $|A| = 0$.

LEMMA 1. *If $\mathfrak{R}$ has a countable cofinal subset and $\mathcal{C}$ is the set of all cofinal subsets of $\mathfrak{R}$, then $|\mathcal{C}| = 1$.*

Let R' be a countable cofinal subset of $\mathfrak{R}$ and suppose $E \notin \mathcal{C}$; then there exists r in R such that $(r)^* \cap E$ is empty. Since there exists r' in R' such that $r' \geqq r$, it follows that $(r')^* \cap E$ is empty. Since r' has an infinite number of distinct successors in R', the set $A_{r'} = \{E \mid (r')^* \cap E \text{ is empty}\}$ is of measure zero. Since $\mathcal{E} - \mathcal{C} = \bigcup_{r' \in R'} A_{r'}$, $|\mathcal{E} - \mathcal{C}| = 0$ so $|\mathcal{C}| = 1$.

By means of this lemma we can define a measure in $\mathcal{C}$ by taking the measure in $\mathcal{E}$ of elements of $\mathcal{C}$; since $|\mathcal{C}| = 1$, we can talk meaningfully about almost all cofinal subsets of $\mathfrak{R}$ [5, Theorem 1.1]. Note that cofinality of E is not affected by adding or removing a finite set, so $\mathcal{C}$ is a "homogeneous" subset of $\mathcal{E}$ and therefore if it is measurable must have measure 0 or 1; which case occurs when $\mathfrak{R}$ does not have a countable cofinal subset, I do not know.

X is a *neighborhood space* [3] if for each x in X is defined a non-

empty family of subsets of X, the neighborhoods of x. If g is a function defined on an index system $\mathfrak{R}$ with values in a neighborhood space X, x is a *limit point* of g (symbol: $x=\lim_{(R,\geqq)} g$) if for each neighborhood N of x and every r_0 in R there exists $r_1\geqq r_0$ such that $g(r)\in N$ whenever $r\geqq r_1$. (This definition is due to Alaoglu and Birkhoff [2]; in case $\mathfrak{R}$ is directed it reduces to the standard simpler form: $x=\lim_{(R,\geqq)} g$ if for each N there exists r_N in R such that $g(r)\in N$ whenever $r\geqq r_N$. $\mathfrak{R}$ is directed if every pair of elements has a common successor.) A point x is called a *cluster point* of g if for every neighborhood N of x and every r_0 in R there exists $r_1\geqq r_0$ such that $g(r_1)\in N$. Clearly every limit point of g is a cluster point of g, but not conversely. (See Lemma 2 below.) If g is a function from $\mathfrak{R}$ into X and E is a cofinal subset of $\mathfrak{R}$, let g_E be the function g reduced onto E and let Qg_E be the set of cluster points of g_E; the function g_E and the set Qg_E will play a role here analogous to that played by the subsequence x' and the set Px' in [1]. Clearly $x=\lim_{(R,\geqq)} g$ implies $x=\lim_{(E,\geqq)} g_E$ for every E cofinal in $\mathfrak{R}$. Let Pg_E be the set of limit points of functions $g_{E'}$ for E' cofinal in E; that is, $x\in Pg_E$ if and only if there exists E' cofinal in $(E, \geqq)$ such that $x=\lim_{(E',\geqq)} g_{E'}$.

Recall that X is said to satisfy Hausdorff's *first countability condition* if for each x in X there is a countable set $\{N_i\}$ of neighborhoods of x such that each neighborhood of x contains an N_i. The next lemma shows the connection between Qg and Pg.

LEMMA 2. *If $\mathfrak{R}$ has a countable cofinal subsystem, if X satisfies the first countability condition, and if the intersection of each pair of neighborhoods of each point x of X contains a third neighborhood of x, then x is a cluster point of g if and only if there exists E cofinal in $\mathfrak{R}$ such that $x=\lim_{(E,\geqq)} g_E$; that is, $Qg=Pg$.*

$Qg\supset Pg$ with no restriction on $\mathfrak{R}$ or X, for $x=\lim_{(E,\geqq)} g_E$ and N a neighborhood of x imply that if $r\in R$, there exists r_1 in E with $r_1\geqq r$ and then an r_2 in E such that $r_2\geqq r_1$ and $g(r_2)\in N$. If $\mathfrak{R}$ and X are restricted as above and if x is a cluster point of g, there exists a sequence $\{N_i'\}$ of neighborhoods of x such that each neighborhood of x contains an N_i'. By the other condition there exists a decreasing sequence of such neighborhoods $N_1\supset N_2\supset\cdots\supset N_i\supset\cdots$. Enumerate R in a sequence $\{r_j'\}$; then let r_1 be a point of $g^{-1}(N_1)$ which follows r_1'; let r_2 and r_3 be points of $g^{-1}(N_2)$ which, respectively, follow r_1 and r_2'; let r_4, r_5, r_6 be points of $g^{-1}(N_3)$ which follow r_2, r_3, and r_3', and so on. Then $E=\{r_i\}$ contains a successor of every element of R', so is cofinal in $(R', \geqq)$ and hence cofinal in $\mathfrak{R}$. If N is a neighborhood of x, there is an $N_j\subset N$ and there exists n such that $g(r_i)\in N_j$ if $i\geqq n$.

Since the set of all r which do not precede any r_i, $i<n$, is cofinal in R and contains all successors of each of its elements, its intersection with E is a set of the same sort in E; this shows that E has the desired property; that is, that $x=\lim_{(E,\geq)} g_E$.

Note that no such relation holds for multiple sequences if the cofinal sets of $\mathfrak{R}$ which are used are restricted as in [1] to be product sets.

We used Lemma 1 to show that "almost everywhere" has meaning in $\mathcal{C}$; a simple application of the same proof gives the next result which can be regarded as an extension of the lemma of [1, §3].

LEMMA 3. *If $\mathfrak{R}$ has a countable cofinal subsystem, if E_0 is cofinal in $\mathfrak{R}$, and if $A=\{E \mid E\cap E_0$ is not cofinal in $E_0\}$, then $|A|=0$; that is, almost every E of $\mathcal{C}$ meets E_0 in a set cofinal in $\mathfrak{R}$.*

Let E_1 be a countable subset of E_0 cofinal in $(E_0, \geq)$; then $E\cap E_0$ not cofinal in E_0 means that there exists r_E in E_1 such that $E\cap E_0\cap(r_E)^*$ is empty. For fixed r in E_1 let $A_r=\{E \mid E\cap E_0\cap(r)^*$ is empty$\}$; since $E_0\cap(r)^*$ is infinite, $|A_r|=0$; since $A=\bigcup_{r\in E_1}A_r$, $|A|=0$.

2. Cluster points. We now proceed to the analogues of the theorems of [1].

THEOREM 1. *If $\mathfrak{R}$ is an index system with a countable cofinal subset, if X satisfies the first countability condition, if g is a function from $\mathfrak{R}$ into X, and if $x\in Qg$, then $x\in Qg_E$ for almost every E of $\mathcal{C}$; that is, each cluster point of g is a cluster point of almost every g_E.*

x is a cluster point of g if and only if $g^{-1}(N)$ is cofinal in $\mathfrak{R}$ for every neighborhood N of x. If $\{N_i\}$ is an equivalent sequence of neighborhoods of x, let $A_i=\{E \mid E\cap g^{-1}(N_i)$ is not cofinal in $g^{-1}(N_i)\}$; then, by Lemma 3, $|A_i|=0$. Setting $A=\mathcal{C}-\bigcup_i A_i$, $|A|=|\mathcal{C}|=1$. If $E\in A$ and N is a neighborhood of x, there is an $N_i\subset N$; since $E\notin A_i$, $E\cap g^{-1}(N_i)$ is cofinal in $g^{-1}(N_i)$ and hence cofinal in $\mathfrak{R}$. Since $E\cap g^{-1}(N)\supset E\cap g^{-1}(N_i)$, $E\cap g^{-1}(N)$ is also cofinal in $\mathfrak{R}$ and therefore is cofinal in $(E, \geq)$; that is, if N is a neighborhood of x and $E\in A$, $g^{-1}(N)\cap E$ is cofinal in $(E, \geq)$, that is, x is a cluster point of g_E if $E\in A$.

Limit points have an analogous property.

THEOREM 1′. *If $\mathfrak{R}$ and X satisfy the hypotheses of Theorem* 1, *then $x=\lim_{(R,\geq)} g$ if and only if $x=\lim_{(E,\geq)} g_E$ for almost every E in $\mathcal{C}$.*

If $x=\lim_{(R,\geq)} g$, then $x=\lim_{(E,\geq)} g_E$ for every E in $\mathcal{C}$. If $x\neq\lim_{(R,\geq)} g$,

there exists a neighborhood N of x and an r_0 in $\mathfrak{R}$ such that every $r_1 > r_0$ has a successor $r_2 > r_1$ for which $g(r_2) \notin N$; let $E_0 = \{r \mid g(r) \in X - N$ and $r > r_0\}$; then if E_1 is so chosen that E_0 and E_1 have no common successors and $E_0 \cup E_1$ is cofinal in $\mathfrak{R}$, by Lemma 3 the set $A = \{E \mid E \cap (E_0 \cup E_1)$ is cofinal in $\mathfrak{R}\}$ is of measure 1. For any such E, $E \cap E_0$ is cofinal in $(E_0, \geqq)$ so $x \neq \lim_{(E, \geqq)} g_E$ if $E \in A$; that is, $x \neq \lim_{(R, \geqq)} g$ implies $x \neq \lim_{(E, \geqq)} g_E$ for almost every E in $\mathcal{C}$.

Say that g is *divergent* if $x = \lim_{(R, \geqq)} g$ is false for every x in X.

COROLLARY. *Let X and $\mathfrak{R}$ satisfy the conditions of the theorem and suppose that g is divergent; then for each x in X the set $A_x = \{E \mid x = \lim_{(E, \geqq)} g_E\}$ is of measure zero. Hence if almost every g_E has a limit point, then Pg is uncountable.*

The first statement follows immediately from the theorem. For the second, $\{E \mid g_E$ has a limit point$\} = \bigcup_{x \in Pg} A_x$; since $|A_x| = 0$ and $|\bigcup_{x \in Pg} A_x| = 1$, Pg is uncountable.

The next two results are related to Theorem 1′ but stronger hypotheses enable us to draw stronger conclusions.

THEOREM 2. *If X and $\mathfrak{R}$ satisfy the conditions of Theorem 1 and if each pair of distinct points of X has a pair of disjoint neighborhoods,*[4] *then g is divergent if and only if almost every g_E is divergent.*

If g has the limit x, so does every g_E. If g is divergent, by the corollary $|A_x| = 0$ for every x. By the first statement in the proof of Lemma 2, if $x_1 = \lim_{(E_1, \geqq)} g_{E_1}$, then x_1 is a cluster point of g; by Theorem 1, x_1 is a cluster point of almost every g_E. Let $A = \{E \mid x_1 \neq \lim_{(E, \geqq)} g_E$ but g_E has a limit point$\}$. If E is in A, let $x = \lim_{(E, \geqq)} g_E$; since there exist disjoint neighborhoods N_1 of x_1 and N of x and since g_E plunges eventually into N, there is an r_1 in E such that $g_E(r) \notin N_1$ if $r > r_1$ and $r \in E$. Hence $g_E^{-1}(N_1)$ is not cofinal in $(E, \geqq)$, so x_1 is not a cluster point of g_E when $E \in A$. Hence $|A| = 0$ by Theorem 1; since $|A_{x_1}| = 0$ also, we see that $|\{E \mid g_E$ has a limit$\}| = |A| + |A_{x_1}| = 0$.

Buck notes that the proof of Theorem 1′ can easily be modified to prove another theorem with the same conclusion as that of Theorem 2.

THEOREM 2′. *If $\mathfrak{R}$ has a countable cofinal subset and if X satisfies Hausdorff's second countability condition,*[5] *then g is divergent if and only if almost every g_E is divergent.*

[4] This is the separation condition in a Hausdorff space; however X need not satisfy the other axioms of such a space.

[5] In this system the second countability condition becomes: There exists a countable subset $\{N_i\}$ of subsets of X such that for each x and each neighborhood N of x there is an i such that N_i contains a neighborhood of x and $N_i \subset N$.

LEMMA 4. *If every neighborhood N of x contains a neighborhood N' of x such that for every y in N' there is a neighborhood N_y of y with $N_y \subset N$, then Qg is closed in X.*

If x is in the closure of Qg, then for every neighborhood N of x there is a point y in $N' \cap Qg$; then for every r_0 in R there exists $r_1 \geqq r_0$ such that $g(r_1) \in N_y \subset N$ so $x \in Qg$.

THEOREM 3. *If $\mathfrak{R}$ and X satisfy the hypotheses of Theorem 1 and Lemma 4, and if Qg is separable, then $Qg = Qg_E$ for almost every E in $\mathfrak{C}$.*

Take a countable dense subset X' of Qg and follow the proof of Theorem 3 of [1], using Theorem 1 and Lemma 4 at the appropriate points. This is much stronger than the corresponding theorem of [1]; the principal extension is that this formulation is valid for all essentially countable index systems rather than for the integers alone. In case $\mathfrak{R}$ is the system of integers, this result includes that of [1] since the hypotheses of [1, Theorem 3] imply the hypotheses of Theorem 1 and Lemmas 2 and 4; Theorems 1 and 2 are not generalizations of the corresponding results of [1] but rather are generalizations in a slightly different direction from the case $n = 1$ of those theorems.

Note that a metric space satisfies all the hypotheses on X except that on Qg in Theorem 3; there the requirement that X is separable would be a sufficient additional condition. Hence with X metric and $\mathfrak{R}$ having a countable cofinal subset, the set Qg used in this paper is equal to the set Pg analogous to Px of [1]. Any countable index system will do for $\mathfrak{R}$ as will the system of real numbers ordered by magnitude or the system of n-tuples of real numbers ordered by $(a_1, \cdots, a_n) \geqq (b_1, \cdots, b_n)$ if $a_i \geqq b_i$ for all $i \leqq n$. Another such system is the system of n-tuples $(r_1, \cdots, r_n)$ where $r_i \in \mathfrak{R}_i$, an index system with a countable cofinal subset, and where $(r_1, \cdots, r_n) > (r_1', \cdots, r_n')$ if $r_1 > r_1'$ or $r_1 = r_1'$ and $r_2 > r_2'$ or, for some $j \leqq n$, $r_i = r_i'$ for $i < j$ while $r_j > r_j'$. (This is the so-called ordinal or lexicographic product of the systems $\mathfrak{R}_i$.) Still another example is the system of pairs of integers where $(i_1, i_2) \geqq (j_1, j_2)$ means that $i_1 = i_2$ and $j_1 \geqq j_2$.

It may be noted by means of Lemmas 1 and 3 that the proofs of Theorems 1 and 2 of [1] also hold when the set $I \times I \times \cdots \times I$ used in [1] as the domain of the function $x = x[i_1, i_2, \cdots, i_n]$ is replaced by $\mathfrak{R}_1 \times \mathfrak{R}_2 \times \cdots \times \mathfrak{R}_n$, where the $\mathfrak{R}_i$ are any index systems with countable cofinal subsystems, providing that $\mathfrak{S}$ is then defined as $\mathfrak{C}_1 \times \mathfrak{C}_2 \times \cdots \times \mathfrak{C}_n$ where $\mathfrak{C}_i$ is the family of cofinal subsets of $\mathfrak{R}_i$. The theorems thus obtained almost include the corresponding results of both papers, the case $n = 1$ giving analogues of the present theorems

with stronger hypotheses, the case where all $\mathfrak{R}_i = I$ giving those of [1].

An open question is whether the existence of a countable cofinal subset is needed to derive the conclusions of Lemmas 1 and 3; if not, some weakening of the hypotheses of the theorems would be possible.

Bibliography

1. R. C. Buck, *Limit points of subsequences*, Bull. Amer. Math. Soc. vol. 50 (1944) pp. 395–397.

2. L. Alaoglu and G. Birkhoff, *Mean ergodic theorems*, Ann. of Math. vol. 41 (1940) pp. 293–309.

3. M. M. Day, *Convergence, closure and neighborhoods*, Duke Math. J. vol. 11 (1944) pp. 181–199.

4. ———, *Oriented systems*, Duke Math. J. vol. 11 (1944) pp. 201–229.

5. J. L. Doob, *Stochastic processes depending on a continuous parameter*, Trans. Amer. Math. Soc. vol. 42 (1937) pp. 107–140.

University of Illinois

A THEOREM ON THE UNIT GROUPS OF SIMPLE ALGEBRAS

RALPH HULL

1. Introduction. Let k be an algebraic number field of finite degree m and let A be a normal simple algebra of degree n, order n^2, over k. Our object is to prove the following theorem.

THEOREM. *If A is an R-algebra, that is, if $n>2$ or at least one infinite prime place of k is unramified in A when $n=2$, then any two distinct maximal orders of A have distinct groups of units.*

There are essential arithmetical differences between algebras which satisfy the R-condition (R-algebras) and those which do not, especially with regard to class-number properties (Eichler [1, 2, 3]).[1] The meaning of the R-condition in the case $n=2$ is as follows. Both k and A are simple algebras over the field k_0 of rational numbers, of orders m and $4m$, respectively, over k_0. Suppose k_0 is extended to the field k_1 of real numbers. Then the extended algebra $k\times k_1$ is the direct sum of fields, each of which is isomorphic either to k_1 or to the field k_2 of complex numbers. This decomposition of $k\times k_1$ involves the decomposition of $A\times k_1$ into a direct sum of simple algebras over k_1, the centers of which are the corresponding summands of $k\times k_1$. Each summand of $A\times k_1$ is either (1) a matrix algebra of degree 2 over k_1, (2) a matrix algebra of degree 2 over k_2, or (3) the division algebra of quaternions over k_1. With each summand of $k\times k_1$ is associated an infinite prime place of k which is said to be ramified or unramified in A according as the corresponding summand of $A\times k_1$ is (3) or is either (1) or (2). The R-condition for $n=2$ is thus equivalent to requiring that not all summands of $A\times k_1$ be (3), in other words, that A over k is not a totally definite quaternion algebra. The condition is in general indispensable in our theorem. For example, the unit groups of all maximal orders in certain definite quaternion algebras over k_0 consist of the units ± 1 only.

The proof of the theorem will be based on the following Hilfssatz due to Eichler [3, p. 239, Hilfssatz 9].

Let A be an R-algebra, let $\mathfrak{O}$ be a maximal order of A, and let $\mathfrak{F}$ be a two-sided $\mathfrak{O}$-ideal. If $\mathcal{A}$ is an element of $\mathfrak{O}$ whose reduced norm $N(\mathcal{A})$ is congruent modulo $\mathfrak{F}$ to a unit of k, then $\mathfrak{O}$ contains a unit $\mathcal{E}\equiv\mathcal{A}$ modulo $\mathfrak{F}$.

Presented to the Society, November 27, 1943; received by the editors December 3, 1943.

[1] Numbers in brackets refer to the references cited at the end of the paper.

By local (p-adic) methods we shall first reduce the proof of the theorem to that for a special case. The lemmas proved for this purpose are independent of the R-condition and may be of interest also in another connection not discussed here, namely, in the problem of imbedding non-maximal orders in maximal orders. Next, also by local methods, we prove the theorem in the special case by applying the Hilfssatz. Finally, we indicate briefly an application of the theorem to the determination of the structure of the Brandt groupoid of normal ideals of A.

2. **Reduction to a special case.** Let A' be a normal simple algebra over a p-adic number field k'. Let ν be the index of A', $n=\nu\kappa$. Then A' is the algebra of matrices of degree κ with elements in a division algebra B' of degree ν over k'. If $\kappa=1$, the unique maximal order $\mathfrak{o}'$ in B' is the only maximal order in $A'=B'$. If $\kappa>1$, the set

$$\mathfrak{O}' = \sum e_{ij}\mathfrak{o}' \qquad (i, j = 1, \cdots, \kappa), \tag{1}$$

where the e_{ij} are ordinary matric units in A', is a maximal order in A'. Conversely, every maximal order in A' is of the form (1) with the appropriate e_{ij}.

Let $\mathfrak{p}'$ be the prime ideal of B'. Then $\mathfrak{p}'$ is a principal ideal: $\mathfrak{p}'=\pi\mathfrak{o}'=\mathfrak{o}'\pi$, where π is a prime element in B'. The ideal $\mathfrak{P}'=\mathfrak{O}'\mathfrak{p}'=\mathfrak{p}'\mathfrak{O}'=\sum e_{ij}\mathfrak{p}'$ is the $\mathfrak{O}'$-prime-ideal in A'. If $\mathfrak{L}'$ is any integral $\mathfrak{O}'$-left-ideal in A', the matric units can be further selected (Hasse [1, p. 524]) so that (1) holds and also

$$\mathfrak{L}' = \mathfrak{O}'\Lambda = \sum e_{ij}\pi^{(a_1+\cdots+a_i)}, \tag{2}$$

where the a's are non-negative integers. The right order of $\mathfrak{L}'$ is

$$\Lambda^{-1}\mathfrak{O}'\Lambda = \sum e_{ij}\mathfrak{p}'^{(a_1+\cdots+a_j)-(a_1+\cdots+a_i)}. \tag{3}$$

The intersection of $\mathfrak{O}'$ and $\Lambda^{-1}\mathfrak{O}'\Lambda$ is

$$\mathfrak{O}' \cap \Lambda^{-1}\mathfrak{O}'\Lambda = \sum_{i\geq j} e_{ij}\mathfrak{o}' + \sum_{i<j} e_{ij}\mathfrak{p}'^{(a_i+1+\cdots+a_j)}. \tag{4}$$

Equations (3) and (4) are readily verified by displaying matrices in the usual way.

With these preliminaries we are ready to prove the following lemma.

LEMMA 1. *Let A' be a normal simple algebra over a p-adic number field k', and let $\mathfrak{O}_1'$ and $\mathfrak{O}_2'$ be any two distinct maximal orders in A', whose distance is $\mathfrak{D}_{12}'=(\mathfrak{O}_2'\mathfrak{O}_1')^{-1}$. Then, either $\mathfrak{D}_{12}'$ is irreducible, or there exists a maximal order $\mathfrak{O}_3'$ in A' such that $\mathfrak{O}_1'\cap\mathfrak{O}_3'\supset\mathfrak{O}_1'\cap\mathfrak{O}_2'$ and such that $\mathfrak{D}_{13}'=(\mathfrak{O}_3'\mathfrak{O}_1')^{-1}$ is irreducible and divides $\mathfrak{D}_{12}'=\mathfrak{D}_{13}'\mathfrak{D}_{32}'$.*

By a remark above $\kappa>1$, since otherwise A' would have only one maximal order. Let $\mathfrak{O}_1'=\mathfrak{O}'$ in (1) and $\mathfrak{D}_{12}'=\mathfrak{L}'$ in (3). Then $\mathfrak{L}'\neq\mathfrak{O}'$, since $\mathfrak{O}_2'\neq\mathfrak{O}'$, and hence at least one a_i in (2) is not zero. If a_1 were not zero, $\mathfrak{L}'$ would be divisible by the two-sided ideal $\mathfrak{P}'$, contrary to the properties of a distance ideal.[2] It follows that there is a fixed $r\geqq 2$ for which $a_1=a_2=\cdots=a_{r-1}=0$, $a_r\neq 0$. If $r=\kappa$, $a_r=1$, $\mathfrak{D}_{12}'=\mathfrak{L}'$ is irreducible and the first alternative in Lemma 1 holds.

Assume that $r<\kappa$ or $a_r>1$ if $r=\kappa$. Then $\mathfrak{L}'$ has the irreducible left divisor

$$\mathfrak{J}_r' = \mathfrak{O}'\Lambda_r, \qquad \Lambda_r = 1 + (\pi - 1)e_{rr}. \tag{5}$$

For, $\Lambda=\Lambda_r\Lambda_0$, where Λ_0 is in $\mathfrak{O}'$. Writing $\mathfrak{O}_2'=\Lambda_r^{-1}\mathfrak{O}\Lambda_r$, $\mathfrak{O}_2'\Lambda_0=\mathfrak{D}'$, we have $\mathfrak{L}'=\mathfrak{J}_r'\mathfrak{D}'$, $\mathfrak{D}'=\mathfrak{O}_2'\Lambda_0$. Since $\mathfrak{J}_r'$ is irreducible and has $\mathfrak{O}'$ and $\mathfrak{O}_2'$ as its left and right orders, $\mathfrak{J}_r'$ is the distance $\mathfrak{D}_{12}'$. It can be shown easily, but is not required in what follows, that $\mathfrak{D}'=(\mathfrak{O}_2'\mathfrak{O}_3')^{-1}=\mathfrak{D}_{32}'$. By a simple computation, we obtain

$$\mathfrak{O}_2' = \Lambda_r^{-1}\mathfrak{O}'\Lambda_r = \sum_{i,j\neq r} e_{ij}\mathfrak{o}' + \sum_{j<r} e_{rj}\mathfrak{p}'^{-1} + \sum_{i<r} e_{ir}\mathfrak{p}' + e_{rr}\mathfrak{o}', \tag{6}$$

and

$$\mathfrak{O}'\cap\mathfrak{O}_2' = \sum_{j\neq r} e_{ij}\mathfrak{o}' + e_{rr}\mathfrak{o}' + \sum_{i\neq r} e_{ir}\mathfrak{p}'. \tag{7}$$

Comparison of (4) and (7) shows that $\mathfrak{O}'\cap\mathfrak{O}_2'\supset\mathfrak{O}'\cap\mathfrak{O}_2'$. This completes the proof of Lemma 1.

The reduction in the large, corresponding to that in Lemma 1, is based on the following lemma.

LEMMA 2. *Let A be a normal simple algebra over an algebraic number field k and let $\mathfrak{O}_1$ and $\mathfrak{O}_2$ be any two distinct maximal orders in A, whose distance is $\mathfrak{D}_{12}=(\mathfrak{O}_2\mathfrak{O}_1)^{-1}$. Let $\mathfrak{p}$ be a prime ideal of k which divides the reduced norm $N(\mathfrak{D}_{12})$. Then, either $\mathfrak{D}_{12}$ is irreducible, in which case $N(\mathfrak{D}_{12})=\mathfrak{p}$, or there exists a maximal order $\mathfrak{O}_3$ in A such that $\mathfrak{O}_1\cap\mathfrak{O}_2\supset\mathfrak{O}_1\cap\mathfrak{O}_2$ and such that $\mathfrak{D}_{13}=(\mathfrak{O}_3\mathfrak{O}_1)^{-1}$ is irreducible, has norm $\mathfrak{p}$, and divides $\mathfrak{D}_{12}=\mathfrak{D}_{13}\mathfrak{D}_{32}$.*

In proving Lemma 2, we write $\mathfrak{O}$ for $\mathfrak{O}_1$ and use primed symbols, without subscripts, to refer to $\mathfrak{p}$-components for a fixed $\mathfrak{p}$ which divides $N(\mathfrak{D}_{12})$.

The ideal $\mathfrak{O}\mathfrak{p}$ is a power of a two-sided $\mathfrak{O}$-prime-ideal P:

$$\mathfrak{O}\mathfrak{p} = \mathfrak{p}\mathfrak{O} = P^n, \qquad n = \nu\kappa, \qquad N(P) = \mathfrak{p}^\kappa, \tag{8}$$

[2] See Deuring [1, chap. VI, VII] for definitions and properties when explicit reference elsewhere is not made.

where $\nu = \nu_p$ and $\kappa = \kappa_p$ are the order of ramification and the capacity, respectively, of p or of P in A. In A, P decomposes into a product of κ irreducible ideals, each of norm p. In accord with (8), the index of A', over k', is ν. The p-component $\mathfrak{D}'_{12}$ is the distance $(\mathfrak{O}'_2 \mathfrak{O}'_1)$ (Hasse [1, p. 529]; cf. Deuring [1, p. 104]). Moreover, $\mathfrak{D}'_{12} \neq \mathfrak{O}'$, since $p \mid N(\mathfrak{D}_{12})$. Hence $\mathfrak{O}'_1 \neq \mathfrak{O}'$, from which it follows that $\kappa \geqq 2$, $\nu < n$. If $\mathfrak{D}_{12}$ is irreducible its norm is necessarily p, and we have nothing to prove. We now assume that $\mathfrak{D}_{12}$ is not irreducible and obtain $\mathfrak{D}_{13}$ and $\mathfrak{O}_3$ of the lemma by the usual method of specifying their components at all finite prime places of k.

Let q_1 denote any prime ideal of k, other than p, which divides $N(\mathfrak{D}_{12})$. If there is no q_1 we omit (10) and the q_1-components in (13), $\cdots$, (16), below, but the argument is not essentially altered. Let q_2 be any prime ideal of k which does not divide $N(\mathfrak{D}_{12})$. By Lemma 1, we have

$$\mathfrak{D}'_{12} = \mathfrak{D}'_{13}\mathfrak{D}'_{32}, \qquad \mathfrak{O}'_1 \cap \mathfrak{O}'_3 \supset \mathfrak{O}' \cap \mathfrak{O}'_2, \tag{9}$$

where $\mathfrak{D}'_{13}$ is the distance $(\mathfrak{O}'_3 \mathfrak{O}'_1)^{-1}$ and is irreducible. Also, for any q_1 and any q_2,

$$\mathfrak{D}'_{12q_1} = \mathfrak{O}'_{q_1}\mathfrak{D}'_{12q_1}, \qquad \mathfrak{O}'_{q_1} \cap \mathfrak{O}'_{q_1} \supset \mathfrak{O}'_{q_1} \cap \mathfrak{O}'_{2q_1}, \tag{10}$$

and

$$\mathfrak{D}'_{12q_2} = \mathfrak{O}'_{q_2}\mathfrak{O}'_{q_2}, \qquad \mathfrak{O}'_{q_2} \cap \mathfrak{O}'_{q_2} = \mathfrak{O}'_{q_2} \cap \mathfrak{O}'_{q_2}, \tag{11}$$

where the first statement in (11) follows from

$$\mathfrak{O}'_{q_2} = \mathfrak{O}'_{2q_2} = \mathfrak{D}'_{12q_2}, \qquad q_2 \nmid N(\mathfrak{D}_{12}). \tag{12}$$

Consider the sets

$$\mathfrak{D}'_{13}, \ \mathfrak{O}'_{q_1}, \ \mathfrak{O}'_{q_2}, \quad \textit{for all } q_1 \textit{ and all } q_2, \tag{13}$$

$$\mathfrak{O}', \ \mathfrak{O}'_{q_1}, \ \mathfrak{O}'_{q_2}, \quad \textit{for all } q_1 \textit{ and all } q_2, \tag{14}$$

$$\mathfrak{O}'_2, \ \mathfrak{O}'_{2q_1}, \mathfrak{O}'_{2q_2}, \quad \textit{for all } q_1 \textit{ and all } q_2, \tag{15}$$

and

$$\mathfrak{O}'_3, \ \mathfrak{O}'_{q_1}, \ \mathfrak{O}'_{q_2}, \quad \textit{for all } q_1 \textit{ and all } q_2. \tag{16}$$

The intersection of (13) and A is an irreducible left divisor $\mathfrak{D}_{13}$ of $\mathfrak{D}_{12}$, that of (14) and A is $\mathfrak{O}$, that of (15) and A is $\mathfrak{O}_2$, and that of (16) and A is $\mathfrak{O}_3$, the right order of $\mathfrak{D}_{13}$. The irreducibility of $\mathfrak{D}_{13}$ follows at once from its definition by (13) and the irreducibility of $\mathfrak{D}'_{13}$. That $\mathfrak{D}_{13}$ divides $\mathfrak{D}_{12}$ follows from the first statements in (9), (10) and (11).

The statements concerning (14), (15) and (16) are evident. Moreover, $\mathfrak{D}_{13}=(\mathfrak{O}_3\mathfrak{O}_1)^{-1}$ since $\mathfrak{D}_{13}$ is irreducible and $\mathfrak{O}_1$ and $\mathfrak{O}_3$ are its left and right orders, respectively. We write $\mathfrak{D}_{12}=\mathfrak{D}_{13}\mathfrak{D}_{32}$ but omit the proof that $\mathfrak{D}_{32}=(\mathfrak{O}_2\mathfrak{O}_3)^{-1}$.

Let $\mathcal{A}$ be any element of $\mathfrak{O}\cap\mathfrak{O}_2$. Then $\mathcal{A}$ is in the p-, q_1-, and q_2-components of both $\mathfrak{O}$ and $\mathfrak{O}_2$, for all q_1 and all q_2. By the second statements in (9), (10) and (11), $\mathcal{A}$ is in the corresponding components of $\mathfrak{O}_3$. Hence $\mathcal{A}$ is in $\mathfrak{O}_3$ and $\mathfrak{O}\cap\mathfrak{O}_3$, since $\mathcal{A}$ is in A. This proves that $\mathfrak{O}\cap\mathfrak{O}_3\supset\mathfrak{O}\cap\mathfrak{O}_2$, and completes the proof of Lemma 2.

3. **Proof of the theorem.** By virtue of Lemma 2, in order to prove the theorem it evidently suffices to prove that *if* $\mathfrak{D}_{13}=(\mathfrak{O}_3\mathfrak{O}_1)^{-1}$ *is irreducible, there is a unit in* $\mathfrak{O}_1$ *which is not in* $\mathfrak{O}_3$.

Let $N(\mathfrak{D}_{13})=p$, and take p to be the fixed prime ideal of k in §2. As before, we use primed symbols, without subscripts, to refer to p-components, and we write $\mathfrak{O}$ for $\mathfrak{O}_1$. We employ the representation (1) of $\mathfrak{O}'$, and have $\kappa\geqq 2$ by an argument used in the proof of Lemma 2. The "canonical" form of $\mathfrak{D}'_{13}$, corresponding to (2), is

$$\mathfrak{D}'_{13}=\mathfrak{O}'\Delta, \qquad \Delta=1+(\pi-1)e_{\kappa\kappa}, \tag{17}$$

and we assume that the e_{ij} are chosen so that (17) holds. The ideal $\mathfrak{D}'_{13}$ divides the two-sided ideal $\mathfrak{O}'P=\mathfrak{P}'$, where P is defined in (8).

Consider the element $\mathcal{A}_p=1+e_{\kappa-1,\kappa}$ of $\mathfrak{O}'$. The reduced norm of an element of $\mathfrak{O}'$ is the determinant of the matrix representing it in (1), with the understanding[3] that the coefficients, in B', are replaced by their corresponding matrices in a reduced representation of B'. It follows that $N(\mathcal{A}_p)=1\equiv 1 \pmod{\mathfrak{O}'P}$. Since $\mathfrak{O}/P$ and $\mathfrak{O}'/\mathfrak{O}'P$ are isomorphic, there exists an element $\mathcal{A}$ in $\mathfrak{O}$ such that $\mathcal{A}\equiv\mathcal{A}_p \pmod{\mathfrak{O}'P}$ and $N(\mathcal{A})\equiv 1 \pmod{P}$. Using, at last, the assumption that A *is an* R*-algebra*, by Eichler's Hilfssatz, $\mathfrak{O}$ contains a unit $\mathcal{E}\equiv\mathcal{A} \pmod{P}$. We shall prove that $\mathcal{E}$ is not in $\mathfrak{O}_3$.

Using (17), we get

$$\mathfrak{O}'_3=\Delta^{-1}\mathfrak{O}'\Delta=\sum_{i,j<\kappa}e_{ij}\mathfrak{o}'+e_{\kappa\kappa}\mathfrak{o}'+\sum_{j<\kappa}e_{\kappa j}\mathfrak{p}'^{-1}+\sum_{i<\kappa}e_{i\kappa}\mathfrak{p}'. \tag{18}$$

Comparison of (1) and (18) shows that $\mathcal{A}_p$ is not in $\mathfrak{O}'_3$. Since $\mathcal{E}\equiv\mathcal{A}_p \pmod{\mathfrak{O}'P}$, $\mathcal{E}$ is not in $\mathfrak{O}'_3$ and hence $\mathcal{E}$ is not in $\mathfrak{O}_3$.

4. **An application of the theorem.** The author was led to the theorem in attempting to answer the following question proposed to him some time ago by Professor R. Baer. The normal ideals of an algebra

[3] This is essential if $\nu>1$, since then B is noncommutative.

A form an infinite groupoid (Brandt [1]) with respect to proper multiplication. The normal ideals having the same left order form, similarly, a *Mischgruppe* (Loewy [1]). Baer [1] has shown that the theory of the latter, and hence the closely related theory of groupoids, is essentially subordinate to the theory of a group G and a non-invariant subgroup H. Indeed, with one trivial exception, every Mischgruppe is isomorphic to the factor-Mischgruppe $(G\|H)_L$ of cosets $H\mathcal{G}$, $\mathcal{G}\in G$, for suitable G and H. Similarly, every Brandt groupoid, with the corresponding exception, is a factor-groupoid $(G\|H)_B$ of cosets $\mathcal{G}_1^{-1}H\mathcal{G}_2$, $\mathcal{G}_i\in G$. The question is: *Are there groups G and H in an algebra A such that $(G\|H)_B$ and $(G\|H)_L$ are isomorphic to the groupoid and Mischgruppe, respectively, of normal ideals, described above?*

By means of our theorem a partial answer to this question can be given. Let $\mathfrak{O}$ be any order of A, U the unit group of $\mathfrak{O}$, A^* the multiplicative group of nonsingular elements of A. Then *if A is an R-algebra, the Mischgruppe of principal $\mathfrak{O}$-left-ideals $\mathfrak{O}\mathcal{A}$, $\mathcal{A}\in A^*$, and the groupoid of ideals $\mathcal{A}_1^{-1}\mathfrak{O}\mathcal{A}_2$, $\mathcal{A}_i\in A^*$ are isomorphic to $(A^*\|U)_L$ and $(A^*\|U)_B$, respectively.*

The Mischgruppe and groupoid described are natural generalizations of the group of principal ideals of k, and hence the above isomorphism, for R-algebras, is a natural generalization of the isomorphism of the group of principal ideals of k with the factor group k^*/u, where u is the group of units of k. However, Baer's formal subordination of the theory of the Brandt groupoid to the theory of groups does not seem to make the concept of a groupoid, as such, any less essential to a satisfactory arithmetical theory of simple algebras. The above isomorphism for R-algebras evidently fails for the rational definite quaternion algebras cited in the introduction since $(A^*\|U)_L$ and $(A^*\|U)_B$ are both groups if U consists of ± 1 only.

References

R. Baer

1. *Zur Einordnung der Theorie der Mischgruppen in die Gruppentheorie*, Sitzungsberichte der Heidelberger Akademie der Wissenschaften, Math.-Naturw. Klasse vol. 4 (1928) pp. 1–18.

H. Brandt

1. *Über eine Verallgemeinerung des Gruppenbegriffes*. Math. Ann. vol. 96 (1927) pp. 360–366.

M. Deuring

1. *Algebren*, Ergebnisse der Mathematik, vol. 4 (1935).

M. Eichler

1. *Bestimmung der Idealklassenzahl in gewissen normalen einfachen Algebren*, Journal für Mathematik vol. 176 (1937) pp. 192–202.

2. *Über die Idealklassenzahl hyperkomplexer Systeme*, Math. Zeit. vol. 43 (1938) pp. 481–494.
3. *Allgemeine Kongruenzklasseneinteilungen der Ideale einfacher Algebren über algebraische Zahlkörpern und ihre L-Reihen*, Journal für Mathematik vol. 179 (1938) pp. 227–251.

H. Hasse
1. *Über p-adische Schiefkörper und ihre Bedeutung für die Arithmetik hyperkomplexer Zahlsysteme*, Math. Ann. vol. 104 (1931) pp. 495–534.

A. Loewy
1. *Über abstrakt definierte Transmutationssysteme oder Mischgruppen*, Journal für Mathematik vol. 157 (1927) pp. 239–254.

The University of Nebraska

ON STRONG SUMMABILITY OF A FOURIER SERIES

FU TRAING WANG

Let $s_n(x)=a/2\sum_{\nu=1}(a_\nu \cos \nu x+b_\nu \sin \nu x)$ be the partial sum of the Fourier series of an integrable periodic function $f(t)$ of period 2π, and let $\phi(t)=\{f(x+t)+f(x-t)-2s\}/2$. We shall establish the following result (Hardy-Littlewood [1][1]).

THEOREM. *If*

$$\int_0^t |\phi(u)| \{1+\log^+ |\phi(u)|\} du = o(t), \qquad \text{as } t\to 0, \tag{1}$$

then $\sum_{\nu=0}^n |s_\nu(x)-s|^2=o(n \log\log n)$, *as* $n\to\infty$.

To prove this theorem we require the following lemmas.

LEMMA 1. *If*

$$\int_0^t |\phi(u)|\, du = o(t), \qquad \text{as } t\to 0, \tag{2}$$

then

$$\sum_{\nu=0}^n |s_\nu(x)-s|^2=\frac{1}{\pi^2}\int_{1/n}^\delta \frac{\phi(t)}{t^2}dt\int_{1/n}^t \phi(u)\frac{\sin n(u-t)}{u-t}du+o(n).$$

PROOF. By (2), for $\nu\leqq n$,

$$s_\nu(x)-s=\frac{2}{\pi}\int_{1/n}^\delta \phi(t)\frac{\sin \nu t}{t}dt+o(1).$$

Hence

$$\begin{aligned}\sum_{\nu=0}^n |s_\nu(x)-s|^2 &= \frac{4}{\pi^2}\int_{1/n}^\delta\int_{1/n}^\delta \frac{\phi(u)\phi(t)}{ut}\left\{\sum_{\nu=1}^n \sin \nu t \sin \nu u\right\}dudt+o(n)\\ &= \frac{2}{\pi^2}\int_{1/n}^\delta\int_{1/n}^\delta \frac{\phi(u)\phi(t)}{ut}\frac{\sin n(u-t)}{u-t}dudt \qquad (3)\\ &\quad +\frac{2}{\pi^2}\int_{1/n}^\delta\int_{1/n}^\delta \frac{\phi(u)\phi(t)}{ut}\frac{\sin n(u+t)}{u+t}dudt+o(n)\\ &= J_1+J_2+o(n).\end{aligned}$$

Received by the editors December 15, 1943.

[1] Numbers in brackets refer to the references listed at the end of the paper.

Set $\Phi(t)=\int_0^t|\phi(u)|\,du$. Then

$$J_2=O\left\{\int_{1/n}^{\delta}\frac{|\phi(t)|}{t^2}dt\int_{1/n}^{t}\frac{|\phi(u)|}{u}du\right\}.$$

But

$$\int_{1/n}^{\delta}\frac{|\phi(t)|}{t^2}dt\int_{1/n}^{\delta}\frac{|\phi(u)|}{u}du=O\left\{\int_{1/n}^{\delta}\frac{|\phi(t)|}{t^2}\log ntdt\right\},$$

and

$$\begin{aligned}\int_{1/n}^{\delta}\frac{|\phi(t)|}{t^2}\log ntdt&=\left[\frac{\Phi(t)}{t^2}\log nt\right]_{1/n}^{\delta}+2\int_{1/n}^{\delta}\frac{\Phi(t)}{t^3}\log ntdt\\&\quad-\int_{1/n}^{\delta}\frac{\Phi(t)}{t^3}dt=o(n)+o\left(n\int_1^{\delta n}\frac{\log t}{t^2}dt\right)\\&\quad+o\left(\int_{1/n}^{\delta}\frac{dt}{t^2}dt\right)=o(n).\end{aligned}$$

Hence

$$J_2=o(n). \tag{4}$$

By inversion of the order of integration, and the resolution of $1/t(t-u)$ into partial fractions, we get

$$\begin{aligned}J_1&=\frac{2}{\pi^2}\int_{1/n}^{\delta}\frac{\phi(t)}{t^2}dt\int_{1/n}^{t}\phi(u)\frac{\sin n(n-t)}{u-t}du\\&\quad+O\left(\int_{1/n}^{\delta}\frac{|\phi(t)|}{t^2}dt\int_{1/n}^{t}\frac{|\phi(u)|}{u}du\right)\\&=\frac{1}{\pi^2}\int_{1/n}^{\delta}\frac{\phi(t)}{t^2}dt\int_{1/n}^{t}\phi(u)\frac{\sin n(u-t)}{u-t}du+o(n).\end{aligned} \tag{5}$$

Lemma 1 follows from (4) and (5).

LEMMA 2. *If $P(u)$ and $F(u)$ are non-negative integrable functions in (a, b), and $\psi(x)$ is a convex function in $(0, \infty)$,*

$$\psi\left\{\int_a^b P(u)F(u)du\Big/\int_a^b P(u)du\right\}\leqq\int_a^b P(u)\psi\{F(u)\}du\Big/\int_a^b P(u)du.$$

This is the well known Jensen inequality (Hardy-Littlewood-Pólya [2]).

PROOF OF THE THEOREM. Since $1+\log^+|\phi(u)| \geqq 1$,

$$\int_0^t |\phi(u)|\,du \leqq \int_0^t |\phi(u)|\,\{1+\log^+|\phi(u)|\}\,du = o(t).$$

Hence by Lemma 1 it suffices to prove that

$$G_n = \int_{1/n}^{\delta} \frac{\phi(t)}{t^2}\,dt \int_{1/n}^{t} \phi(u)\,\frac{\sin n(u-t)}{u-t}\,du = o(n \log\log n).$$

Now

$$\begin{aligned} G_n &= \int_{1/n}^{n^{-1}\log n} \frac{\phi(t)}{t^2}\,dt \int_{1/n}^{t} \phi(u)\,\frac{\sin n(u-t)}{u-t}\,du \\ &\quad + \int_{n^{-1}\log n}^{\delta} \frac{\phi(t)}{t^2}\,dt \int_{1/n}^{t-n^{-1}\log n} \phi(u)\,\frac{\sin n(u-t)}{u-t}\,du \\ &\quad + \int_{n^{-1}\log n}^{\delta} \frac{\phi(t)}{t^2}\,dt \int_{t-n^{-1}\log n}^{t} \phi(u)\,\frac{\sin n(u-t)}{u-t}\,du \\ &= I_1 + I_2 + I_3. \end{aligned} \tag{6}$$

By (2) we have

$$I_1 = \int_{1/n}^{n^{-1}\log n} \frac{|\phi(t)|}{t^2}\,O(nt)\,dt = o(n \log\log n), \tag{7}$$

and

$$I_2 = \int_{n^{-1}\log n}^{\delta} \frac{|\phi(t)|}{t^2}\,O\left(\frac{n}{\log n}\,t\right)dt = o(n). \tag{8}$$

Now set

$$P(u) = \left|\frac{\sin n(u-t)}{u-t}\right|,$$

$$k = \int_{t-n^{-1}\log n}^{t} P(u)\,du \sim \log\log n,$$

and

$$J_n(t) = \int_{t-n^{-1}\log n}^{t} |\phi(u)|\,P(u)\,du.$$

Applying Lemma 2 with $\psi(x) = x(1+\log^+ x)$, we find

$$\psi\left\{\frac{J_n(t)}{k}\right\} \leqq \frac{\int_{t-n^{-1}\log n}^{t} |\phi(u)|\,\{1+\log^+|\phi(u)|\}\,P(u)\,du}{k} \leqq \frac{nt}{k}.$$

Hence

$$J_n(t) \leqq nt\left\{1 + \log^+ \frac{J_n(t)}{k}\right\}^{-1}.$$

If we suppose now that

$$J_n(t) \geqq A_1 nt/\log ntk^{-1}, \tag{9}$$

we find $1+\log^+ (J_n(t)/k) \geqq A \log (nt/k)$, for $nt/k \geqq \log n/\log\log n$. Hence

$$J_n(t) \leqq A_2 nt/\log ntk^{-1}, \tag{10}$$

for all values of n and t such that (9) and the relation $nt \geqq \log n$ hold true.

If (9) does not hold,

$$J_n(t) \leqq A_1 nt/\log ntk^{-1}. \tag{11}$$

By (10) and (11) we have then

$$J_n(t) \leqq A_2 nt/\log ntk^{-1} \tag{12}$$

for all values of n and t for which $nt \geqq \log n$.

By (12)

$$\begin{aligned} |I_3| &\leqq \int_{n^{-1}\log n}^{\delta} \frac{|\phi(t)|}{t^2} J_n(t)dt \leqq A\int_{n^{-1}\log n}^{\delta} \frac{|\phi(t)|}{t^2} \frac{n}{\log (n/\log\log n)} dt \\ &\leqq An\left\{\left[\frac{\Phi(t)}{t} \frac{1}{\log (nt/\log\log n)}\right]_{n^{-1}\log n}^{\delta}\right. \\ &\quad + \int_{n^{-1}\log n}^{\delta} \frac{\Phi(t)}{t^2} \frac{dt}{\log (nt/\log\log n)} \\ &\quad \left. + \int_{n^{-1}\log n}^{\delta} \frac{\Phi(t)}{t^2} \frac{dt}{(\log (nt/\log\log n))^2}\right\} \\ &= o(n\log\log n). \end{aligned}$$

From this inequality and (6), (7), and (8) we draw

$$G_n = o(n\log\log n).$$

In conclusion I remark that by the above method I established in a recent paper the following result, which is a generalization of Hardy and Littlewood's theorem [1].

If

$$\int_0^t |\phi(u)|\,du = O(t) \quad \text{and} \quad \int_0^t \phi(u)du = o(t) \quad \text{as } t \to 0,$$

then

$$\sum_{\nu=0}^{n} | s_\nu(x) - s |^2 = o(n \log n), \qquad \text{as } n \to \infty.$$

References

1. G. H. Hardy and J. E. Littlewood, *On strong summability for a Fourier series*, Fund. Math. vol. 25 (1935).

2. G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge, 1934, p. 151.

National University of Chekiang

A NOTE ON RIESZ SUMMABILITY OF THE TYPE e^{n^α}

FU TRAING WANG

Recently I proved the following result in the case $r=2$ (Wang [4][1]).

Let $\sigma_n^{(r)}$ be the rth Cesàro mean of the series $\sum_{n=0}^{\infty} a_n$. If $\sigma_n^{(r)} - s = o(n^{-r\alpha})$ $0<\alpha<1$, as $n\to\infty$, where r is a positive integer, and $a_n > -Kn^{\alpha-1}$, the series converges to sum s.

For the case $r=1$ this result has been established by Boas [1]. His argument, however, does not seem to be applicable in any simple way to the general case.

The object of this note is to prove a theorem on Riesz summability of type e^{n^α}, and then to deduce the result above from a Tauberian theorem of Hardy [2].

Let us put $C_r(\omega) = a_0 e^{r\omega^\alpha} + \sum_{n<\omega} (e^{\omega^\alpha} - e^{n^\alpha})^r a_n$. A series $\sum_{n=0}^{\infty} a_n$ is said to be summable (e^{n^α}, τ) to the sum s if

$$C_r(\omega) = se^{r\omega^\alpha} + o(e^{r\omega^\alpha}). \tag{1}$$

The result by Hardy which is to be called upon is the following: If the series $\sum_{n=0}^{\infty} a_n$, with terms $a_n \geqq -Kn^{\alpha-1}$, $0<\alpha<1$, is summable (e^{n^α}, τ), it is convergent. We shall now prove the following theorem.

THEOREM. *If $\sigma_n^{(r)} - s = o(n^{-r\alpha})$, $0<\alpha<1$, as $n\to\infty$, the series $\sum_{n=0}^{\infty} a_n$ is summable (e^{n^α}, τ) to the sum s, where $\tau > r/(1-\alpha)$.*

To prove this let $\beta_n = (e^{\omega^\alpha} - e^{n^\alpha})^r$, $\Delta\beta_n = \beta_n - \beta_{n+1}$, $\Delta^{r+1}\beta_n = \Delta^r\beta_n - \Delta^r\beta_{n+1}$ and

$$s_n^{(r)} = \sum_{\nu=0}^{n} \binom{n-\nu+r}{n-\nu} a_\nu,$$

$m = [\omega]$. Then, by successive Abel's transformations we have

$$\begin{aligned} C_r(\omega) &= a_0 e^{r\omega^\alpha} + \sum_{n=1}^{m} \beta_n a_n \\ &= a_0 e^{r\omega^\alpha} + \sum_{n=1}^{m-r+1} s_n^{(r)} \Delta^{r+1}\beta_n + \sum_{i=0}^{r} s_{m-i}^{(i)} \Delta^i \beta_{m-i} - \sum_{i=0}^{r} s_0^{(i)} \Delta^i \beta_1 \\ &= a_0 e^{r\omega^\alpha} + J_1 + J_2 - J_3. \end{aligned} \tag{2}$$

Received by the editors January 6, 1944.

[1] Numbers in brackets refer to the references listed at the end of the paper.

Since $\beta_{m-i} = (e^{\omega^\alpha} - e^{(m-i)^\alpha})^r = O(e^{r\omega^\alpha}\omega^{r(\alpha-1)})$, it follows that

$$\Delta^{(i)}\beta_{m-i} = O(e^{r\omega^\alpha}\omega^{r(\alpha-1)}) \qquad \text{for } 0 \leqq i \leqq r.$$

By a familiar theorem on Cesàro sums we get

$$s^{(i)}_{m-i} = O(\omega^r), \qquad \text{for } 0 \leqq i \leqq r,$$

and from this

$$J_2 = O(e^{r\omega^\alpha}\omega^{r(\alpha-1)+r}) = o(e^{r\omega^\alpha}). \tag{3}$$

Since $\Delta^i\beta_1 = O(e^{r\omega^\alpha}\omega^{i(\alpha-1)})$, for $1 \leqq i \leqq r$, and $\beta_1 = (e^{\omega^\alpha} - e)^r$, $s_0^{(i)} = s_0 = a_0$ we get

$$J_3 = e^{r\omega^\alpha}a_0 + o(e^{r\omega^\alpha}). \tag{4}$$

By the hypothesis of the theorem we have

$$J_1 = \sum_{n=1}^{m-r-1} s_n^{(r)}\Delta^{r+1}\beta_n = s\sum_{n=1}^{m-r-1}\binom{n+r}{n}\Delta^{r+1}\beta_n + o\left(\sum_{n=1}^{m-r-1} n^{r(1-\alpha)}\,|\Delta^{r+1}\beta_n|\right). \tag{5}$$

It follows by mathematical induction that

$$\Delta^{r+1}\beta_n = (-1)^{r+1}\int_n^{n+1} dx_1 \int_{x_1}^{x_1+1} dx_2 \cdots \int_{x_r}^{x_r+1} B^{(r+1)}(x_{r+1})\,dx_{r+1},$$

where

$$\beta^{(r+1)}(x) = \frac{d^{r+1}}{dx^{r+1}}\left\{(e^{\omega^\alpha} - e^{x^\alpha})^r\right\}.$$

By direct differentiation we have

$$B^{(r+1)}(x) = \sum_{j=1}^{r+1}\sum_{i=1}^{\tau} c_{ij}e^{(r-j)\omega^\alpha}e^{ix^\alpha}x^{i\alpha-r-1},$$

where τ is taken as a positive integer and the constants c_{ij} depend upon i, j, τ, r, α. Hence we get

$$\Delta^{r+1}\beta_n = \sum_{i=1}^{\tau} O(e^{(r-i)\omega^\alpha}e^{in^\alpha}n^{(r+1)(\alpha-1)}). \tag{6}$$

It is easily verified by Abel's transformation that

$$\sum_{n=1}^{m-r-1}\binom{n+r}{n}\Delta^{r+1}\beta_n = e^{r\omega^\alpha} + o(e^{r\omega^\alpha}). \tag{7}$$

Hence by (5), (6), and (7)

$$J_1 = se^{\tau\omega^\alpha} + o(e^{\tau\omega^\alpha}) + o\left(\sum_{i=1}^{r}\sum_{n=1}^{m-r-1} e^{(\tau-\delta)\omega^\alpha} e^{i\eta^\alpha} \eta^{\alpha-1}\right) \tag{8}$$

$$= se^{\tau\omega^\alpha} + o(e^{\tau\omega^\alpha}).$$

The proof of the theorem follows from (4), (2), (3), and (8).

I conclude by observing that the theorem is the best possible of its kind (Wang [4]).

References

1. R. P. Boas, *Tauberian theorem for* (C.1) *summability*. Duke Math. J. vol. 4 (1938) pp. 227–230.

2. G. H. Hardy, *An extension of a theorem on oscillating series*, Proc. London Math. Soc. (2) vol. 12 (1913) pp. 174–180.

3. G. H. Hardy and M. Riesz, *The general theory of Dirichlet series*, 1914.

4. F. T. Wang, *Some remarks of oscillating series*, Quarterly J. Math.

National University of Chekiang

ON RIESZ SUMMABILITY OF FOURIER SERIES BY EXPONENTIAL MEANS

FU TRAING WANG

Let $f(t)$ be an integrable periodic function with the period 2π. Let its Fourier series be

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt \tag{1}$$

and let

$$\phi(t) = \{f(x+t) + f(x-t) - 2s\}/2,$$

$$\phi_\beta(t) = (1/\Gamma(\beta)) \int_0^t (t-u)^{\beta-1}\phi(u)du,$$

$$A_n = a_n \cos nx + b_n \sin nx.$$

We shall prove the following result.[1]

If $A_n > -Kn^{-\beta/\gamma}$ $(\gamma > \beta > 0)$ *and*

$$\phi_\beta(t) = o(t^\gamma) \qquad \text{as } t \to 0, \tag{2}$$

the Fourier series (1) *converges to* s *at* $t=x$.

Set $\alpha = 1-\beta/\gamma$, and

$$C_\tau(\omega) = a_0 e^{\tau\omega^\alpha}/2 + \sum_{n<\omega} (e^{\omega^\alpha} - e^{n^\alpha})^\tau A_n. \tag{3}$$

The Fourier series (1) is said to be summable (e^{n^α}, τ) to the sum s if[2]

$$C_\tau(\omega) = se^{\tau\omega^\alpha} + o(e^{\tau\omega^\alpha}) \qquad \text{as } \omega \to \infty.$$

Concerning this kind of summability we have the following theorem.

THEOREM.[3] *If* (2) *holds and* τ *is a positive integer greater than* $\gamma+1$ *the Fourier series* (1) *is summable* (e^{n^α}, τ) *to the sum* s *at* $t=x$.

Received by the editors January 6, 1944.

[1] G. H. Hardy and J. E. Littlewood [2], F. T. Wang [6]. Numbers in brackets refer to the references listed at the end of the paper.

[2] G. H. Hardy and M. Riesz [3].

[3] Under the hypotheses of the Theorem I have established that the Fourier series (1) is summable $(e^{n^\alpha}, \gamma+\delta)(\delta>0)$ to the sum s at $t=x$, but the proof is very complicated. See F. T. Wang [6].

The convergence criterion above is deducible from this theorem by the use of a result of Hardy,[4] namely *if the series* $\sum_{n=0}^{\infty} a_n$, *with terms* $a_n \geqq -Kn^{\alpha-1}$, $0<\alpha<1$, *is summable* (e^{n^α}, τ), *it is convergent.*

To prove the theorem we write

$$E_\tau(\omega, t) = \tau\alpha \int_0^\omega (e^{\omega^\alpha} - e^{x^\alpha})^{\tau-1} e^{x^\alpha} x^{\alpha-1} \frac{\sin xt}{t}\, dx.$$

Then we have[5] $C_\tau(\omega) = (2/\pi)\int_0^1 \phi(t) E_\tau(\omega, t)dt + se^{\tau\omega^\alpha} + o(e^{\tau\omega^\alpha})$, and if we take $\omega_1 = 2^{-1/\alpha}\omega$, then

$$\begin{aligned} E_\tau(\omega, t) &= \alpha\tau \int_{\omega_1}^\omega (e^{\omega^\alpha} - e^{x^\alpha})^{\tau-1} e^{x^\alpha} x^{\alpha-1} \frac{\sin xt}{t}\, dx + o(e^{\tau\omega^\alpha}) \\ &= F_\omega(t) + o(e^{\tau\omega^\alpha}). \end{aligned}$$

Hence

$$C_\tau(\omega) = (2/\pi) \int_0^1 \phi(t) F_\omega(t)dt + se^{\tau\omega^\alpha} + o(e^{\tau\omega^\alpha}). \tag{4}$$

By setting $n = [\beta]+1$ and differentiating under the integral sign we get

$$\begin{aligned} F_\omega^{(n)}(t) &= \frac{d^n}{dt^n}\left\{\alpha\tau \int_{\omega_1}^\omega (e^{\omega^\alpha} - e^{x^\alpha})^{\tau-1} e^{x^\alpha} x^{\alpha-1} \frac{\sin xt}{t}\, dx\right\} \\ &= \sum_{i=0}^{n} K_i \int_{\omega_1}^\omega (e^{\omega^\alpha} - e^{x^\alpha})^{\tau-1} e^{x^\alpha} x^{\alpha-1+n-i} \frac{\sin (xt - a)}{t^{i+1}}\, dx, \end{aligned} \tag{5}$$

[6]

where $a = (n-i)\pi/2$.

By mathematical induction we can easily establish the formula

$$\begin{aligned} &(d^\tau/dx^\tau)\left\{(e^{\omega^\alpha} - e^{x^\alpha})^{\tau-1} e^{x^\alpha} x^{\alpha-1+n-i}\right\} \\ &\qquad = \sum_{j=1}^{\tau} \sum_{p=0}^{\tau} K_{jp} e^{(\tau-j)\omega^\alpha} e^{jx^\alpha} x^{(p+1)(\alpha-1)+n-i-(\tau-p)}. \end{aligned} \tag{6}$$

Then by the use of (5) and (6) and an integration by parts we find

$$\begin{aligned} F_\omega^{(n)}(t) &= \sum_{i=0}^{n} \sum_{j=1}^{\tau} \sum_{p=0}^{\tau-1} K_{ijp} e^{(\tau-j)\omega^\alpha} \int_{\omega_1}^\omega e^{jx^\alpha} x^c \frac{\sin (xt - b)}{t^{i+\tau}}\, dx \\ &\quad + O(e^{(\tau-1/2)\omega^\alpha} \omega^{k_1 t - k_2}), \end{aligned} \tag{7}$$

[4] G. H. Hardy [4].

[5] F. T. Wang [6].

[6] Throughout this paper we use K or $K_i \cdots$ as a constant different in different occurrences.

where $c=(p+1)(\alpha-1)+n-i-(\tau-1-p)$, and $b=(n-i-\tau+1)\pi/2$.

Put $t=1$ in (7). Then $F_\omega^{(m)}(0)$ is finite for $1\leqq m<n$, and $F_\omega^{(m)}(1)=o(e^{\tau\omega^\alpha})$. Successive integration of (4) by parts yields

$$C_\tau(\omega)=(2/\pi)\int_0^1 \phi_n(t\,F_\omega^{(n)}(t)dt+se^{\tau\omega^\alpha}+o(e^{\tau\omega^\alpha}). \tag{8}$$

By a theorem on the fractional integral[7]

$$\phi_n(t)=(1/\Gamma(n-\beta))\int_0^t (t-u)^{n-\beta-1}\phi_\beta(u)du,$$

we have

$$C_\tau(\omega)=(2/\pi)\int_0^1 \phi_\beta(u)H_\omega(u)du+se^{\tau\omega^\alpha}+o(e^{\tau\omega^\alpha}), \tag{9}$$

where

$$\begin{aligned} H_\omega(u)&=(1/\Gamma(n-\beta))\int_u^1 (t-u)^{n-\beta-1}F_\omega^{(n)}(t)dt \quad (n>\beta>n-1)\\ &=F_\omega^{(n)}(u) \qquad\qquad (n=\beta). \end{aligned} \tag{10}$$

Concerning $H_\omega(u)$ we require the following two lemmas.

LEMMA 1. *For $\omega>K$ and $0<u<1$,*

$$\begin{aligned} H_\omega(u)=&\sum_{i=0}^{n-1}O(e^{\tau\omega^\alpha}\omega^{\beta-i}u^{-i-1})+\sum_{i=0}^{n-1}O(e^{\tau\omega^\alpha}\omega^{n-i-1}u^{n-\beta-i-2})\\ &+O(e^{\tau\omega^\alpha}\omega^{n-1}(1-u)^{n-\beta-1})+O(e^{\tau\omega^\alpha}u^{-\beta-1}). \end{aligned}$$

PROOF. From (10) and (5) we have

$$\begin{aligned} H_\omega(u)=&\sum_{i=0}^{n}K_i\int_{\omega_1}^{\omega}(e^{\omega^\alpha}-e^{x^\alpha})^{\tau-1}e^{x^\alpha}x^{\alpha-1+n-i}dx\\ &\cdot\int_u^1(t-u)^{n-\beta-1}\frac{\sin(xt-a)}{t^{i+1}}dt. \end{aligned} \tag{11}$$

Now

$$\int_1^\infty (t-u)^{n-\beta-1}\frac{\sin(xt-a)}{t^{i+1}}dt=O\{(1-u)^{n-\beta-1}x^{-1}\}. \tag{12}$$

[7] L. S. Bosanquet [1].

It follows from a change of variable, the second mean value theorem, and a theorem on the Γ function,[8] that

$$\begin{aligned}(13)\qquad &\int_u^\infty (t-u)^{n-\beta-1}\frac{\sin(xt-a)}{t^{i+1}}\,dt\\ &= u^{n-\beta-i-1}\int_0^\infty v^{n-\beta-1}\sin(x^a(1+v)-a)\,dv + O(u^{n-\beta-i-2}x^{-1})\\ &= O(u^{-i-1}x^{\beta-n}) + O(u^{n-\beta-i-2}x^{-1}).\end{aligned}$$

The lemma is proved by (11), (12), (13) and an easy estimate of the term $i=n$ in (11).

LEMMA 2. *For $\omega>K$ and $0<u<1$,*

$$\begin{aligned}H_\omega(u) &= \sum_{i=0}^{n} O(e^{\tau\omega^a}u^{-\tau-i-1}\omega^{\tau(a-1)+\beta-i}) + O(e^{(\tau-1/2)\omega^a}\omega^{k_1}u^{-k_2})\\ &\quad + \sum_{i=0}^{n} O(e^{\tau\omega^a}u^{n-\beta-\tau-i-1}\omega^{(\tau-1)(a-1)+n-i-1})\\ &\quad + O(e^{\tau\omega^a}\omega^{(\tau-1)(a-1)+n-1}(1-u)^{n-\beta-1}).\end{aligned}$$

PROOF. From (13) we get

$$\begin{aligned}(14)\qquad H_\omega(u) &= \sum_{i=0}^{n}\sum_{j=1}^{\tau}\sum_{p=0}^{\tau-1} K_{ijp}e^{(\tau-j)\omega^a}\int_{\omega_1}^{\omega} e^{ix^a}x^c\,dx\\ &\quad\cdot\int_u^1 (t-u)^{n-\beta-1}\frac{\sin(xt-b)}{t^{i+\tau}}\,dt + O(e^{(\tau-1/2)\omega^a}\omega^{k_1}u^{-k_2}),\end{aligned}$$

and

$$\begin{aligned}(15)\qquad &\int_u^1 (t-u)^{n-\beta-1}\frac{\sin(xt-b)}{t^{i+\tau}}\,dt\\ &\quad = u^{n-\beta-i-\tau}\int_0^\infty v^{n-\beta-1}\sin\{xu(1+v)-b\}\,dv\\ &\qquad + O((1-u)^{n-\beta-1}x^{-1}) + O(u^{n-\beta-i-\tau-1}x^{-1})\\ &\quad = u^{-\tau-i}x^{\beta-n}\Gamma(\beta-n)\sin(xu-b')\\ &\qquad + O((1-u)^{n-\beta-1}x^{-1}) + O(u^{n-\beta-i-\tau-1}x^{-1}).\end{aligned}$$

From (14) and (15), Lemma 2 follows.

[8] E. C. Titchmarsh [5, p. 107].

PROOF OF THE THEOREM. By Lemma 1 and (2)

$$\int_0^{\omega^{\alpha-1}} \phi_\beta(u) H_\omega(u) du = o(e^{\tau\omega^\alpha}) \qquad \text{as } \omega \to \infty, \tag{16}$$

and by (2) and Lemma 2

$$\int_{\omega^{\alpha-1}}^1 \phi_\beta(u) H_\omega(u) du = o(e^{\tau\omega^\alpha}) \qquad \text{as } \omega \to \infty. \tag{17}$$

By (9), (16), and (17), then,

$$C_\tau(\omega) = se^{\tau\omega^\alpha} + o(e^{\tau\omega^\alpha}) \qquad \text{as } \omega \to \infty.$$

Thus the theorem is proved.

REFERENCES

1. L. S. Bosanquet, *On Abel's integral equation and fractional integral*, Proc. London Math. Soc. (2) vol. 31 (1930) pp. 134–143.

2. G. H. Hardy and J. E. Littlewood, *New convergence criteria for a Fourier series*, Annali Scuola Normale Superiore, Pisa (2) vol. 3 (1934) pp. 43–62.

3. G. H. Hardy and M. Riesz, *The general theory of Dirichlet series*, 1912, Cambridge.

4. G. H. Hardy, *An extension of a theorem on oscillating series*, Proc. London Math. Soc. (2) vol. 12 (1913) pp. 174–180.

5. E. C. Titchmarsh. *The theory of functions*, 1932.

6. F. T. Wang, *On Riesz summability of Fourier series*, Proc. London Math. Soc. (2) vol. 47 (1942) pp. 308–325.

NATIONAL UNIVERSITY OF CHEKIANG

ON THE ASYMPTOTIC EXPANSIONS OF ENTIRE FUNCTIONS DEFINED BY MACLAURIN SERIES

H. K. HUGHES

1. Introduction. Let $f(z)$ be an entire function defined by a Maclaurin series of the form

$$f(z) = \sum_{n=0}^{\infty} g(n) z^n \tag{1}$$

with infinite radius of convergence. We wish, in this paper, to make some observations of a general nature relative to the asymptotic behavior of this function in the neighborhood of the point at infinity. Some restrictions will, of course, be placed on the coefficient $g(n)$ of z^n.

To begin with, we shall suppose $g(n)$ to be of such character that we can replace n by the complex variable $w=x+iy$. Then we shall assume that the resulting function $g(w)$ satisfies the following two conditions when it is considered throughout any arbitrary right half-plane $x>x_0$:

(a) it is single-valued and analytic;

(b) corresponding to any positive number ϵ, there exists a positive constant K, dependent only on ϵ and x_0, such that

$$|g(x+iy)| < Ke^{(\gamma+\epsilon)|y|} \tag{2}$$

where γ is a non-negative constant.

Under these conditions, if k is a positive integer such that $k \geqq \gamma$, then it is known that we may express $f(z)$ in the form

$$f(z) = \int_{-l-1/2}^{\infty} g(x)[(-1)^{k+1}z]^x \frac{\sin k\pi x}{\sin \pi x}\, dx - \sum_{n=1}^{l} g(-n)z^{-n} + \xi(z, l), \tag{3}$$

where l is any positive integer, and where if $|\arg [(-1)^{k+1}z]| < \pi$, we have $\lim_{|z|\to\infty} z^l \xi(z, l) = 0$.

This theorem was first proved for the special case in which $k=1$ by W. B. Ford [1, p. 30].[1] The general form (3) is due to C. V. Newsom [4]. Both Ford and Newsom have used their results in finding the asymptotic expansions of several special types of functions [1, chap-

Presented to the Society, April 28, 1944; received by the editors January 7, 1944.

[1] Numbers in square brackets refer to references at the end of the paper.

ters 6, 7; 5; 6]. The theorem was also used in [2]. A study of the examples appearing in these investigations has suggested to the writer the generalities which are noted in the following sections.

2. **A fundamental lemma.** It will be observed that equation (3) does not furnish complete information regarding the asymptotic expansions of the function $f(z)$, due to the presence of the integral in the right-hand member. By finding the asymptotic expansions of this integral under suitable restrictions on $g(w)$, we shall arrive at a general theorem concerning such expansions of $f(z)$.

The proof of the theorem rests upon a lemma regarding the asymptotic character of a certain definite integral. This lemma is as follows:

LEMMA. *Consider the function*

$$F(z) = \int_{-\lambda}^{\infty} H(x)z^x dx, \tag{4}$$

where λ is a sufficiently large positive number.[2] *Suppose that the function $H(w)$, where $w=x+iy$, is single-valued and analytic in the finite w-plane and, when considered throughout the arbitrary right half-plane $x>x_0$, can be expressed in the form*

$$H(w) = \frac{c_0}{\Gamma(w+1)} + \frac{c_1}{\Gamma(w+2)} + \cdots + \frac{c_s + \delta(w, s)}{\Gamma(w+s+1)},$$

where the c's are constants, and $\lim_{|w|\to\infty} \delta(w, s)=0$; $s=0, 1, 2, \cdots$. Then for values of z of large modulus such that $|\arg z|<\pi$, we have

$$F(z) \sim \begin{cases} e^z \sum_{n=0}^{\infty} c_n z^{-n}; & |\arg z| \leqq \pi/2, \\ 0; & |\arg z| > \pi/2. \end{cases} \tag{5}$$

The proof of this lemma is omitted here since it requires only slight changes in one of the discussions appearing in the work of Ford [1, art. 20].

3. **A general theorem.** We proceed now to study the function $f(z)$ defined by the series (1). We shall establish the following general theorem regarding its asymptotic expansions.

THEOREM. *Given the entire function*

[2] The result also holds if λ is a complex number whose real part is positive and sufficiently large. The integration then takes place from $-\lambda$ to $+\infty$ along the line parallel to the real axis.

$$f(z) = \sum_{n=0}^{\infty} g(n)z^n, \tag{6}$$

suppose that the function $g(w)$, *where* $w=x+iy$, *is single-valued and analytic in the finite w-plane and, when considered throughout the arbitrary right half-plane* $x>x_0$, *can be expressed in the form*

$$g(w) = \sigma^w\left\{\frac{c_0}{\Gamma(\alpha w+t)} + \frac{c_1}{\Gamma(\alpha w+t+1)} + \cdots + \frac{c_s+\delta(\alpha w, s)}{\Gamma(\alpha w+t+s)}\right\}, \tag{7}$$

where σ *and* α *are positive, the* c*'s and* t *are constants, and* $\lim_{|w|\to\infty}\delta(\alpha w, s)=0$; $s=0, 1, 2, \cdots$. *Then, when considered for values of* z *of large modulus such that* $-\pi<\arg z\leqq\pi$, *the function* $f(z)$ *is developable asymptotically in the form*

$$f(z) \sim \frac{1}{\alpha}\sum_{\mu}\left\{Y_\mu^{(1-t)}e^{Y_\mu}\sum_{n=0}^{\infty}c_nY_\mu^{-n}\right\} - \sum_{n=1}^{\infty}g(-n)z^{-n}, \tag{8}$$

where Y_μ *denotes the expression* $\sigma^{1/\alpha}z^{1/\alpha}e^{2\pi i\mu/\alpha}$, *and the symbol* $\sum_\mu$ *denotes summation over those integral values of* μ *which satisfy* $|\arg z+2\pi\mu|\leqq\pi\alpha/2$.

Proof. Under the conditions postulated, when $x>x_0$, we may write

$$|g(x+iy)| < Ke^{(\pi\alpha/2+\epsilon)|y|}$$

where ϵ and K have the meanings described in §1.[3] Consequently, the theorem of Ford and Newsom is applicable to the function $f(z)$ if k is properly chosen. If a proper choice be made, then, for the corresponding range of arg z, we have

$$f(z) = \phi(z) - \sum_{n=1}^{l} g(-n)z^{-n} + \xi(z, l), \tag{9}$$

where $\phi(z)$ denotes the integral

$$\int_{-l-1/2}^{\infty} g(x)[(-1)^{k+1}z]^x\,\frac{\sin k\pi x}{\sin \pi x}\,dx.$$

In particular, if k is selected as the smallest odd integer $2p+1$ such that $k\geqq\alpha/2$, then the factor $[(-1)^{k+1}z]^x$ in the integrand of $\phi(z)$ becomes z^x, and (9) is then valid when $|\arg z|<\pi$. On the other hand, if k is taken as $2p+2$, then this factor becomes $(-z)^x$, and (9) holds for $0<\arg z=\pi+\arg(-z)<2\pi$, and hence in particular when $\arg z=\pi$.

[3] For the special property of the gamma function from which this inequality results, see [1, p. 61].

Now we have

$$\frac{\sin k\pi x}{\sin \pi x} = \begin{cases} \sum_{\mu=-p}^{p} e^{2\pi i\mu x}; & k = 2p+1, \\ e^{\pi i x} \sum_{\mu=-p-1}^{p} e^{2\pi i\mu x}; & k = 2p+2. \end{cases}$$

And since $e^{\pi i z}\cdot(-z)^z = z^z$, it follows that for either of the two values of k mentioned, the integral $\phi(z)$ is equal to the sum of k integrals of the form

$$h_\mu(z) = \int_{-l-1/2}^{\infty} g(x)(z\cdot e^{2\pi i\mu})^z dx, \tag{10}$$

where we now have $-\pi < \arg z \leqq \pi$, and the integer μ takes on the values indicated in the above identities.

We shall next obtain the asymptotic expansions of the integral in (10). First, we recall that $g(x)$ is expressible in form (7). Hence, if we change the variable of integration in (10) from x to x' through the transformation $\alpha x + t = x' + 1$, and then drop the primes, (10) assumes the form

$$h_\mu(z) = \frac{1}{\alpha} Y_\mu^{(1-t)} \int_{-\lambda}^{\infty} H(x) Y_\mu^x dx, \tag{11}$$

where Y_μ is as already defined, $\lambda = \alpha(l+1/2)+1-t$, and $H(x)$ is of the form

$$\frac{c_0}{\Gamma(x+1)} + \frac{c_1}{\Gamma(x+2)} + \cdots + \frac{c_s + \delta(x, s)}{\Gamma(x+s+1)}.$$

We may therefore apply the fundamental lemma of §2 to the integral appearing in (11), assuming that l has been chosen sufficiently large. In (5), z is to be replaced by Y_μ, and arg z by $(\arg z + 2\pi\mu)/\alpha$. Thus, for values of z such that $-\pi < \arg z \leqq \pi$ and $|\arg z + 2\pi\mu| < \pi\alpha$, we have

$$h_\mu(z) \sim \begin{cases} \frac{1}{\alpha} Y_\mu^{(1-t)} e^{Y_\mu} \sum_{n=0}^{\infty} c_n (Y_\mu)^{-n}; \\ 0, \end{cases} \tag{12}$$

the first result holding when $|\arg z + 2\pi\mu| \leqq \pi\alpha/2$, and the second otherwise.

Having established (12), we have only to sum the expansions of

$h_\mu(z)$ over the proper values of μ in order to arrive at the expansion of the integral $\phi(z)$. Evidently this will yield the expansion

$$\phi(z) \sim \frac{1}{\alpha} \sum_{\mu} \left(Y_\mu^{(1-\ell)} e^{Y_\mu} \sum_{n=0}^{\infty} c_n Y_\mu^{-n} \right), \tag{13}$$

where $\sum_\mu$ has the meaning already stated.

Finally, if we replace $\phi(z)$ in (9) by its asymptotic expansion given in (13), and write the result in asymptotic notation, we arrive at (8). This concludes the proof of the theorem.

4. Special cases. Dominant terms. We shall now note the simplified forms assumed by (8) for certain values of α and of $\arg z$. At the same time, we shall pick out the dominant terms in each form. First, if α is such that $0<\alpha<2$, then the condition $|\arg z+2\pi\mu| \leqq \pi\alpha/2$ is satisfied only by $\mu=0$ when $|\arg z| \leqq \pi\alpha/2$. Hence for such α and $\arg z$, we have

$$f(z) \sim \frac{1}{\alpha} (\sigma z)^{(1-\ell)/\alpha} \exp (\sigma z)^{1/\alpha} \sum_{n=0}^{\infty} c_n (\sigma z)^{-n/\alpha} - \sum_{n=1}^{\infty} g(-n) z^{-n}. \tag{14}$$

Here we see that the first series is dominant when $\arg z$ is confined to the narrower range $|\arg z| <\pi\alpha/2$, but not when $|\arg z| =\pi\alpha/2$. Furthermore, if $\alpha=2$, and $|\arg z| <\pi$, expansion (14) still holds, and the first series is dominant.

On the other hand, if we have $0<\alpha<2$ and $|\arg z| >\pi\alpha/2$, but still such that $-\pi <\arg z \leqq \pi$, then no integral value of μ satisfies $|\arg z+2\pi\mu| \leqq \pi\alpha/2$. Hence, for such values of z, (8) reduces to the form

$$f(z) \sim - \sum_{n=1}^{\infty} g(-n) z^{-n}. \tag{15}$$

It is to be noted that this result is consistent with another theorem due to Ford [1, p. 4] according to which (15) holds throughout the open sector $\pi\alpha/2 <\arg z <(2-\alpha/2)\pi$.

Finally, we note that when $\alpha \geqq 2$, the terms affected by $\sum_\mu$ are dominant over the last series in (8), this being true for all values of $\arg z$ under consideration.

5. Supplementary remarks. Several supplementary observations relative to the theorem established in §3 are now to be noted. In the first place, we have required that the function $g(w)$ have no finite singularities. Suppose, however, that it does have a singularity at $w=w_0$, while (7) continues to hold for all values of $|w|$ sufficiently

large. Then if w_0 is not a negative integer, the theorem still holds provided that we subtract from the right member of (8) the asymptotic expansion of the loop integral

$$\frac{1}{2i}\int_C \frac{g(w)[(-1)^{k+1}z]^w}{e^{ki\pi w}\sin \pi w}\,dw. \tag{16}$$

The loop C surrounds the point w_0 and extends to infinity in any convenient direction lying in either the third or the fourth quadrant. If w_0 is a negative integer, say $-m$, then the term $g(-m)z^{-m}$ in (8) is suppressed, and the expansion of (16) is put in its place. For justification of these statements, the reader is referred to [3], where a method for finding the expansion of (16) is also discussed.

Secondly, the theorem which we have here established narrows down the difficulties involved in determining the asymptotic expansion of a given entire function of form (1). For the problem of finding the expansion of the integral appearing in (3) is solved if it can be shown that the function $g(w)$ can be written in the form (7).

Thirdly, the theorem can evidently be applied to a large variety of entire functions. In many instances in which the coefficient $g(n)$ involves the gamma function, relation (7) is satisfied. In this connection, we note that if the value of σ in (7) is unity, then $f(z)$ is necessarily the function

$$F_\alpha(z, t) = \sum_{n=0}^{\infty} \frac{h(n)}{\Gamma(\alpha n + t)} z^n$$

which was discussed in [2]. The writer plans to apply the theorem to certain further important types of functions, and to report the results in later papers.

References

1. W. B. Ford, *The asymptotic developments of functions defined by Maclaurin series*, University of Michigan Studies, Scientific Series, vol. 9, 1936.

2. C. G. Fry and H. K. Hughes, *Asymptotic developments of certain integral functions*, Duke Math. J. vol. 9 (1942) pp. 791–802.

3. H. K. Hughes, *On a theorem of Newsom*, Bull. Amer. Math. Soc. vol. 49 (1943) pp. 288–292.

4. C. V. Newsom, *On the character of certain entire functions in distant portions of the plane*, Amer. J. Math. vol. 60 (1938) pp. 561–572.

5. C. V. Newsom, and A. Franck, *Upon the asymptotic representation of functions of the Bessel type*, Boletin Matematico vol. 14 (1940) pp. 11–14.

6. C. V. Newsom, *The asymptotic behavior of a class of entire functions*, Amer. J. Math. vol. 65 (1943) pp. 450–454.

Purdue University

SCHUR'S THEOREMS ON COMMUTATIVE MATRICES

N. JACOBSON

In 1905 I. Schur[1] proved that the maximum number $N(n)$ of linearly independent commutative matrices of n rows and columns is given by the formula $N(n)=[n^2/4]+1=\nu^2+1$ if $n=2\nu$ and $=\nu(\nu-1)+1$ if $n=2\nu-1$. Schur also determined the sets of linearly independent commutative matrices containing $N(n)$ elements. In this note we give a simpler derivation of Schur's results and an extension of these results from algebraically closed fields to arbitrary fields.

If $A_1, \cdots, A_{N(n)}$ is a set of linearly independent commutative matrices, the set $\mathfrak{A}$ of matrices $\sum A_i\phi_i$ where ϕ_i is arbitrary in the underlying field Φ is a commutative subalgebra containing the identity of the matrix algebra Φ_n. Hence $N(n)$ is the maximal dimensionality of commutative subalgebras of Φ_n. It is easy to see that $N(n)\geqq[n^2/4]+1$. For consider the set $\mathfrak{Z}_n$ of matrices

$$\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} \tag{1}$$

where if $n=2\nu$, A is arbitrary in Φ_ν and if $n=2\nu-1$, A is an arbitrary matrix of ν rows and $\nu-1$ columns. Thus $\dim \mathfrak{Z}_n=[n^2/4]$. It may be verified that $\mathfrak{Z}_n$ is a zero algebra. Hence the algebra $\mathfrak{B}_n$ obtained by adjoining 1 to $\mathfrak{Z}_n$ is a commutative algebra of dimensionality $[n^2/4]+1$. We remark also that if $n=2\nu-1$ we may replace $\mathfrak{Z}_n$ by the algebra $\overline{\mathfrak{Z}}_n$ of matrices of the form (1) in which A is an arbitrary matrix of $\nu-1$ rows and ν columns. We denote by $\overline{\mathfrak{B}}_n$ the extension of $\overline{\mathfrak{Z}}_n$ obtained by adjoining 1.

To prove that $N(n)\leqq[n^2/4]+1$ it suffices to assume that Φ is algebraically closed. For if $A_1, \cdots, A_{N(n)}$ are linearly independent and commutative in Φ_n, then they have these properties in Σ_n for any extension field Σ of the field Φ. Thus $N(n, \Phi)\leqq N(n, \Sigma)$. We shall therefore assume that Φ is algebraically closed. Let $\mathfrak{A}$ be a commutative subalgebra of Φ_n containing the identity and let N be the dimensionality of $\mathfrak{A}$ over Φ. We suppose first that $\mathfrak{A}$ is an indecomposable algebra of matrices. Then it is known that by replacing $\mathfrak{A}$ by a similar set we may suppose that the matrices of $\mathfrak{A}$ have the form

Received by the editors January 14, 1944.

[1] *Zur Theorie vertauschbaren Matrizen*, J. Reine Angew. Math. vol. 130 (1905) pp. 66–76.

$$(2) \qquad \begin{pmatrix} \alpha & & * \\ & \alpha & \\ & & \ddots \\ & & \ddots \\ 0 & & \alpha \end{pmatrix}.$$

Thus $\mathfrak{A}=(1)+\mathfrak{N}$ where $\mathfrak{N}$ is a nilpotent algebra of matrices in proper triangular form, that is, of the form (2) in which $\alpha=0$. Evidently $\dim \mathfrak{N}=N-1$.

Let the k_1th column ($k_1>1$) be the first column for which there exists a matrix U_{1k_1} in $\mathfrak{N}$ with element in the $(1, k_1)$ position not equal to 0. We may suppose that the element in the $(1, k_1)$ position of U_{1k_1} is 1. We normalize U_{1k_1} further by using the following lemma.

LEMMA 1. *Let $U\in\Phi_n$ and let V be the matrix obtained from U by adding the kth column multiplied by θ to the lth column ($k\neq l$) and then subtracting the lth row multiplied by θ from the kth row. Then U and V are similar.*

We have $V=S^{-1}US$ where $S=1+e_{kl}\theta$, e_{kl} the matrix with 1 in the (k, l) position and 0's elsewhere.

We may apply this lemma to U_{1k_1} and replace it by a matrix whose first row is $e_{k_1}=(0, \cdots, 1, 0, \cdots, 0)$ where the 1 is in the k_1th column. The operations required for this purpose are additions of multiples of the k_1th column to later columns and additions to the k_1th row of later rows. These operations replace $\mathfrak{N}$ by a properly triangular set of matrices $\mathfrak{N}'$ similar to $\mathfrak{N}$ such that all the elements in the $(1, j)$ position with $j<k_1$ in $\mathfrak{N}'$ are 0 and such that $\mathfrak{N}'$ contains a matrix V_{1k_1} (similar to U_{1k_1}) whose first row is e_{k_1}. Now let $\mathfrak{P}'$ be the subspace of $\mathfrak{N}'$ of matrices in which the elements in the $(1, k_1)$ position are 0 and suppose that the k_2th column ($k_2>k_1$) is the first column for which there is a matrix U_{1k_2} in $\mathfrak{P}'$ with element in the $(1, k_2)$ place not equal to 0. Evidently any matrix in $\mathfrak{N}'$ has the form $V_{1k_1}\beta_1+P'$, P' in $\mathfrak{P}'$. We now apply to U_{1k_2} the process used before for U_{1k_1} and replace it by a matrix V_{1k_2} similar to it and having e_{k_2} for first row. The set $\mathfrak{N}'$ will be transformed into a set $\mathfrak{N}''$ of properly triangular matrices and V_{1k_1} changed into a new matrix which we shall again denote as V_{1k_1} with first row e_{k_1}. Any matrix in $\mathfrak{N}''$ has the form $A=V_{1k_1}\beta_1+P''$, P'' in $\mathfrak{P}''$, the transform of the set $\mathfrak{P}'$. It is clear that the elements in the $(1, j)$ position, $j<k_2$, for any matrix in $\mathfrak{P}''$ are 0. Hence $A=V_{1k_1}\beta_1+V_{1k_2}\beta_2+S''$ where S'' is in the subspace $\mathfrak{S}''$ of $\mathfrak{N}''$ of matrices having 0 in the $(1, j)$ position with $j\leqq k_2$. This process may be continued and proves the following lemma.

LEMMA 2. *The set $\mathfrak{N}$ is similar to a set $\mathfrak{N}^{(r)}$ of properly triangular*

matrices that contain matrices $V_{1k_1}, \cdots, V_{1k_r}$, *such that the first row of* V_{1k_i} *is* e_{k_i}, $1<k_1<k_2< \cdots <k_r$, *and such that any matrix in* $\mathfrak{N}^{(r)}$ *has the form* $\sum V_{1k_i}\beta_i+Z$, *where* Z *has first row* 0.

Now let $\mathfrak{N}_2$ be the subset of $\mathfrak{N}^{(r)}$ of matrices Z having first row 0. Evidently $\mathfrak{N}^{(r)}=\{V_{1k_1}, \cdots, V_{1k_r}\}+\mathfrak{N}_2$ and the V_{1k_i} are linearly independent. Hence dim $\mathfrak{N}^{(r)}=N-1=r+\dim \mathfrak{N}_2$. Now we note that if $Z\in\mathfrak{N}_2$, the first row of $V_{1k_i}Z$ is the k_ith row of Z and the first row of ZV_{1k_i} is 0. Hence the k_ith row of every matrix Z in $\mathfrak{N}_2$ is 0.

We now repeat the argument for $\mathfrak{N}_2$. Then $\mathfrak{N}_2$ may be replaced by a set $\mathfrak{N}_2^{(s)}$ similar to $\mathfrak{N}_2$ such that (1) $\mathfrak{N}_2^{(s)}$ is properly triangular, (2) $\mathfrak{N}_2^{(s)}$ contains matrices $V_{2l_1}, \cdots, V_{2l_s}$ having first row 0 and second row $e_{l_1}, \cdots, e_{l_s}$, respectively, such that any matrix in $\mathfrak{N}_2^{(s)}$ has the form $\sum V_{2l_i}\beta_i+Z$ where Z is a matrix with first two rows 0. Let $\mathfrak{N}_3$ denote the set of matrices Z. We assert that if $s=l_i$ or $s=k_j$ then the sth row of $\mathfrak{N}_3$ is 0. This is clear if $s=l_i$. Hence suppose that $s=k_j\neq$any l_i. Then the matrices of $\mathfrak{N}_2$ all have k_jth row 0 and the operations performed in passing from $\mathfrak{N}_2$ to $\mathfrak{N}_2^{(s)}$ do not affect this row. Hence the k_jth row of every matrix in $\mathfrak{N}_2^{(s)}$ is 0. Evidently $N-1=r+s+\dim \mathfrak{N}_3$.

We now write $k_i=k_{1i}$, $l_i=k_{2i}$, $r=r_1$, $s=r_2$. Then if we continue this process we see that $N-1$ is equal to the number of matrices in the following set

$$
\begin{array}{c}
e_{1k_{11}}, \cdots, e_{1k_{1r_1}} \\
e_{2k_{21}}, \cdots, e_{2k_{2r_2}} \\
\cdots\cdots\cdots
\end{array} \tag{3}
$$

where $1<k_{11}< \cdots <k_{1r_1}$, $2<k_{21}<k_{22}< \cdots <k_{2r_2}, \cdots$, and $r_i=0$ if $i=k_{jl}$ with $j<i$. Let $s_1, s_2, \cdots, s_m$ be the complete set of integers k_{ij} arranged in increasing order. Then it is clear that $N-1 \leqq N(s_1, s_2, \cdots, s_m)$, the number of matrices in the set

$$
\begin{array}{l}
e_{1s_1}, e_{2s_1}, \cdots, e_{s_1-1,s_1} \\
e_{1s_2}, e_{2s_2}, \cdots, e_{s_1-1,s_2}, e_{s_1+1,s_2}, \cdots, e_{s_2-1,s_2} \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{array} \tag{4}
$$

Evidently

$$
\begin{aligned}
N(s_1, s_2, \cdots, s_m) &= (s_1-1)+(s_2-2)+\cdots+(s_m-m) \\
&= \sum s_i - m(m+1)/2.
\end{aligned} \tag{5}
$$

Hence we have

$$N-1 \leqq N(s_1, \cdots, s_m) \leqq N(n-m+1, \cdots, n) = m(n-m). \tag{6}$$

Now $m(n-m)$ attains its maximum value for $m=[n/2]$. If $n=2\nu$ this maximum is ν^2 and if $n=2\nu-1$, it is $\nu(\nu-1)$. Thus the maximum value is $[n^2/4]$. This proves for indecomposable algebras $\mathfrak{A}$ the following theorem.

THEOREM 1. *If $\mathfrak{A}$ is a commutative subalgebra of Φ_n,* dim $\mathfrak{A} \leqq [n^2/4]+1$.

If $\mathfrak{A}$ is decomposable we suppose that the matrices of $\mathfrak{A}$ have the form

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

where $A \in \Phi_{n_1}$ and $B \in \Phi_{n_2}$, $n_i \geqq 1$, $n_1+n_2=n$. We may assume that the theorem holds for the Φ_{n_i}.

Case 1. $n=2\nu-1$, $n_1=2\nu_1-1$, $n_2=2\nu_2$. Here $\nu=\nu_1+\nu_2$ and $N \leqq \nu_1(\nu_1-1)+1+\nu_2^2+1 \leqq \nu(\nu-1)+1$. Equality holds between the last two terms only when $n=3$.

Case 2. $n=2\nu$, $n_1=2\nu_1-1$, $n_2=2\nu_2-1$. Here $\nu=\nu_1+\nu_2-1$ and $N \leqq \nu_1(\nu_1-1)+1+\nu_2(\nu_2-1)+1 \leqq \nu^2+1$. Equality holds only if $n=2$.

Case 3. $n=2\nu$, $n_1=2\nu_1$, $n_2=2\nu_2$. Here $\nu=\nu_1+\nu_2$ and $N=\nu_1^2+1+\nu_2^2+1<\nu^2+1$. Thus the theorem is proved.

We have also proved the following theorem.

THEOREM 2. *The maximum number $N(n)$ of linearly independent commutative matrices of n rows and columns is given by the formula* $N(n)=[n^2/4]+1$.

We shall investigate next the form of commutative subalgebras $\mathfrak{A}$ of Φ_n of the maximum dimensionality $N(n)$. Suppose first that $\mathfrak{A}$ has the structure $\mathfrak{A}=(1)+\mathfrak{N}$ where $\mathfrak{N}$ is a nilpotent algebra. Then it is known that by replacing $\mathfrak{A}$ by a similar set we may suppose that the matrices of $\mathfrak{N}$ are properly triangular. We may apply the above considerations to $\mathfrak{N}$. By (3), (4), (5) and (6) we see that if $n=2\nu$ we must have $k_{11}=k_{21}=\cdots=k_{\nu 1}=\nu+1, \cdots, k_{1\nu}=k_{2\nu}=\cdots=k_{\nu\nu}=n$ as the set of k's in (3). If $n=2\nu-1$ the set of k's is either $k_{11}=\cdots=k_{\nu 1}=\nu+1, \cdots, k_{1\,\nu-1}=\cdots=k_{\nu\,\nu-1}=n$ or $k_{11}=\cdots=k_{\nu-1\,1}=\nu, \cdots, k_{1\nu}=\cdots=k_{\nu-1\,\nu}=n$. Suppose first that n is even. Let $\mathfrak{N}^{(r)}$ $(r=\nu)$ and $\mathfrak{N}_2$ be determined as before. It is clear that $\mathfrak{N}^{(r)}$ is similar to $\mathfrak{N}$ by a matrix in Φ_n and we need not assume here that Φ is algebraically closed. The matrices of $\mathfrak{N}_2$ have the form

$$B = \left(\begin{array}{c|c} \overbrace{0 \cdots 0}^{\nu} & \overbrace{0 \cdots 0}^{\nu} \\ R & A \\ \hline 0 & 0 \end{array}\right) \begin{matrix} \\ \\ \} \nu \end{matrix}. \tag{7}$$

Since $k_{21} = \nu + 1$ it is clear that the second row of R is 0. Moreover the operations used to pass from $\mathfrak{N}_2$ to $\mathfrak{N}_3$ affect only the last ν rows and last ν columns of $\mathfrak{N}_2$. Hence the third row of R is the same as the third row of the corresponding matrix in $\mathfrak{N}_2$. Since $k_{31} = \nu + 1$ the third row of R is 0. Similarly the other rows of R are 0, and $R = 0$ in (7). Now $\dim \mathfrak{N}_2 = \nu^2 - \nu$. Hence $\mathfrak{N}_2$ consists of all matrices of the form (7) in which $R = 0$ and A is arbitrary. Let

$$V_{1j} = \left(\begin{array}{c|ccccc} 0 & 0 \cdots 0 & 1 & 0 \cdots 0 \\ \hline & & V_j & & \\ \hline & & T_j & & \end{array}\right), \qquad j = \nu + 1, \cdots, n,$$

where the 1 is in the jth column and T_j is a properly triangular matrix. Since $V_{1j}B = BV_{1j}$ the following holds in Φ_ν:

$$\begin{pmatrix} 0 \cdots 0 \\ A \end{pmatrix} T_j = 0.$$

Since A is arbitrary, $T_j = 0$. Thus $\mathfrak{N}^{(\nu)}$ is the set $\mathfrak{Z}_n$ and $\mathfrak{A}$ is similar to the algebra $\mathfrak{B}_n$ defined before. If n is odd a similar argument shows that $\mathfrak{A}$ is similar either to $\mathfrak{B}_n$ or to $\overline{\mathfrak{B}}_n$.

We suppose now that $\mathfrak{A}$ is arbitrary. Evidently $\mathfrak{A}$ contains the identity matrix. Since $n > 3$ by the proof of Theorem 1, $\mathfrak{A}$ is indecomposable. Moreover if Ω is the algebraic closure of Φ then $\mathfrak{A}_\Omega$ is an indecomposable algebra containing the identity. It follows that $\mathfrak{A}_\Omega$ is similar to a set of matrices of the form (1). Hence $\mathfrak{A}_\Omega = (1) + \mathfrak{Z}$ where $\mathfrak{Z}$ is nilpotent and so $\mathfrak{A}_\Omega$ is similar to either $\mathfrak{B}_n(\Omega)$ or $\overline{\mathfrak{B}}_n(\Omega)$. Thus $\mathfrak{Z}$ is a zero algebra. Now let $\mathfrak{N}$ be the radical of the algebra $\mathfrak{A}$ and consider the semi-simple algebra $\overline{\mathfrak{A}} = \mathfrak{A} - \mathfrak{N}$. The extension $\overline{\mathfrak{A}}_\Omega$ is a homomorphic image of $\mathfrak{A}_\Omega$. Hence $\overline{\mathfrak{A}}_\Omega = (1) + \overline{\mathfrak{Z}}$ where $\overline{\mathfrak{Z}}$ is a zero algebra. The structure of $\overline{\mathfrak{A}}$ is given by the following lemma.

LEMMA 3. *If $\overline{\mathfrak{A}}$ is a semi-simple commutative algebra such that $\overline{\mathfrak{A}}_\Omega = (1) + \overline{\mathfrak{Z}}$ where $\overline{\mathfrak{Z}}$ is a zero algebra, then either $\overline{\mathfrak{A}} = (1)$ or Φ is an imperfect field of characteristic 2 and $\overline{\mathfrak{A}} = \Phi(x)$ where $x^2 = \xi$, a non-square in Φ.*

Since $\mathfrak{A}$ is semi-simple, $\overline{\mathfrak{A}}$ is a direct sum of fields, but since $\overline{\mathfrak{A}}_\Omega$ has only one idempotent element, $\overline{\mathfrak{A}}$ is a field. Let $\overline{\mathfrak{A}} > (1)$. Then $\overline{\mathfrak{A}}$ has no

separable subfields, for if Σ were such a subfield Σ_0 is a direct sum of fields and $\overline{\mathfrak{A}}_0$ would contain more than one idempotent element. Thus Φ has characteristic $p \neq 0$ and $\overline{\mathfrak{A}}$ contains an element x such that $x^p = \xi$ is in Φ where ξ is not a pth power in Φ. Now there exists an element η in Ω such that $\eta^p = \xi$ and hence the element $z = x - \eta$ in $\overline{\mathfrak{A}}_0$ is nilpotent of index p. Since $\overline{\mathfrak{Z}}$ is a zero algebra, $p = 2$. It follows readily that in this case $\mathfrak{A} = \Phi(x)$, $x^2 = \xi$.

This lemma shows that unless Φ is an imperfect field of characteristic 2 any commutative subalgebra $\mathfrak{A}$ of $\Phi_n (n > 3)$ of maximum dimensionality has a difference algebra with respect to its radical $\mathfrak{N}$ of dimensionality 1. Since $\mathfrak{A}$ contains the identity, $\mathfrak{A} = (1) + \mathfrak{N}$. As we have seen, this implies that $\mathfrak{A}$ is similar to either $\mathfrak{B}_n$ or to $\overline{\mathfrak{B}}_n$.

THEOREM 3. *Suppose that Φ is not an imperfect field of characteristic 2 and let $n > 3$. Then if $\mathfrak{A}$ is a subalgebra of Φ_n of maximum dimensionality $N(n)$, $\mathfrak{A}$ is similar to $\mathfrak{B}_n$ if $n = 2\nu$ and $\mathfrak{A}$ is similar to either $\mathfrak{B}_n$ or $\overline{\mathfrak{B}}_n$ if $n = 2\nu - 1$.*[2]

As a consequence we have the following theorem.

THEOREM 4. *Let Φ, n and $\mathfrak{A}$ be as in Theorem 3. Then $\mathfrak{A} = (1) + \mathfrak{N}$ where $\mathfrak{N}$ is a zero algebra.*

We remark finally that if n is odd the sets $\mathfrak{B}_n$ and $\overline{\mathfrak{B}}_n$ are not similar. This may be seen by considering the sets $\mathfrak{Z}_n$ and $\overline{\mathfrak{Z}}_n$. Let $\mathfrak{S}(\overline{\mathfrak{S}})$ be the space determined by the columns of the matrices of $\mathfrak{Z}_n(\overline{\mathfrak{Z}}_n)$. Then $\dim \mathfrak{S} = \nu$ and $\dim \overline{\mathfrak{S}} = \nu - 1$. On the other hand if $\mathfrak{Z}_n$ were similar to $\overline{\mathfrak{Z}}_n$ we would have $\dim \mathfrak{S} = \dim \overline{\mathfrak{S}}$. It follows that $\mathfrak{Z}_n$ and $\overline{\mathfrak{Z}}_n$ are not similar and hence $\mathfrak{B}_n$ and $\overline{\mathfrak{B}}_n$ are not similar. Thus in this case there are for $n = 2\nu - 1 > 3$ two distinct classes in the sense of similarity of commutative subalgebras of dimensionality $N(n)$.

JOHNS HOPKINS UNIVERSITY

[2] If $n = 2, 3$, $\mathfrak{A}$ may be decomposable. The determination of these algebras is readily obtained.

PROJECTIVE INVARIANTS OF INTERSECTION OF CERTAIN PAIRS OF SURFACES

CHUAN-CHIH HSIUNG

1. **Introduction.** In a recent paper [2][1] the author has shown the existence, together with metric and projective characterizations,[2] of a unique projective invariant determined by the neighborhood of the second order of two surfaces S_1, S_2 at an ordinary point O in ordinary space when the tangent planes τ_1, τ_2 of the surfaces S_1, S_2 at the point O are distinct and the line t of intersection of the two tangent planes τ_1, τ_2 does not coincide with any one of the asymptotic tangents of the surfaces S_1, S_2 at the point O. On the other hand, with regard to the coincidences of the line t and the asymptotic tangents of the surfaces S_1, S_2 at the point O two essentially different cases can arise. The object of this note is to derive some projective invariants in these cases. Noticing that no projective invariant can be determined by the neighborhood of the second order of the surfaces S_1, S_2 at the point O, we obtain all projective invariants determined by the neighborhood of the second order of one surface and that of the third order of the other at the point O.

I. Two surfaces with distinct tangent planes and distinct asymptotic tangents at an ordinary point

2. **Derivation of invariants.** Let us first consider two surfaces S_1, S_2 in ordinary space intersecting at an ordinary point O with distinct tangent planes τ_1, τ_2, whose line of intersection t is supposed to be an asymptotic tangent of the surface S_1 at the point O. Let t_1 be the other asymptotic tangent of the surface S_1 at the point O, and t_2 the harmonic conjugate line of t with respect to the asymptotic tangents of the surface S_2 at the point O. If we choose the point O to be the origin, the lines t, t_2, t_1 to be respectively the axes x, y, z of a general non-homogeneous projective coordinate system, then the power series expansions of the surfaces S_1, S_2 in the neighborhood of the point O may be written in the form

$$(1) \qquad S_1: \quad y = lxz + px^3 + rx^2z + sxz^2 + qz^3 + \cdots,$$

$$(2) \qquad S_2: \quad z = mx^2 + ny^2 + \cdots.$$

Received by the editors January 26, 1944.

[1] Numbers in square brackets refer to the references at the end of the paper.

[2] An extension of these results to two hypersurfaces has been made by Professor Su. See his paper [6].

In order to find projective invariants of the surfaces S_1, S_2 at the point O, we have to consider the most general projective transformation of coordinates which shall leave the lines t, t_1, t_2 unchanged:

$$
\begin{aligned}
x &= a_{22}x^*/(1 + a_{12}x^* + a_{13}y^* + a_{14}z^*),\\
y &= a_{33}y^*/(1 + a_{12}x^* + a_{13}y^* + a_{14}z^*),\\
z &= a_{44}z^*/(1 + a_{12}x^* + a_{13}y^* + a_{14}z^*),
\end{aligned} \tag{3}
$$

where a_{ik} are arbitrary constants. The effect of this transformation on equations (1), (2) is to produce two other equations of the same form whose coefficients, indicated by stars, are given by the formulas

$$
\begin{aligned}
a_{33}l^* &= a_{22}a_{44}l, \qquad a_{33}p^* = a_{22}^2p, \qquad a_{33}q^* = a_{44}^2q,\\
2a_{12}a_{33}l^* + a_{33}r^* &= a_{12}a_{22}a_{44}l + a_{22}^2a_{44}r,\\
2a_{14}a_{33}l^* + a_{33}s^* &= a_{14}a_{22}a_{44}l + a_{22}a_{44}^2s;\\
a_{44}m^* &= a_{22}^2m, \qquad a_{44}n^* = a_{33}^2n.
\end{aligned} \tag{4}
$$

Further elimination of a_{ik} from equations (4) gives immediately that *the quantities*

$$I = lm/p, \qquad J = p^2q^4m^3/l^{12}n^3 \tag{5}$$

are projective invariants associated with the surfaces S_1, S_2, at the point O.

3. **Projective characterizations of the invariants** I, J. It is well known that the tangent plane τ_1 intersects the surface S_1 in a curve with a node at the point O, the nodal tangents being the asymptotic tangents t, t_1. The expansion of the branch C_1 of this curve which is tangent to the tangent t, $z=y=0$, is easily found to be

$$z = -(p/l)x^2 + \cdots. \tag{6}$$

On the other hand, the curve C_2 in which the tangent plane τ_1 intersects the surface S_2 is given by the equations

$$z = mx^3 + \cdots, \quad y = 0. \tag{7}$$

It is thus easy to reach the conclusion that *the projective invariant I associated with the point O of the surfaces S_1, S_2 is, except for sign, equal to the projective invariant of contact*[3] *of the two plane curves C_1, C_2 at the point O.*

To characterize projectively the other invariant J it is useful to

[3] The projective invariant of contact of two plane curves having ordinary contact at a point was first found by H. J. S. Smith [5] and R. Mehmke [3]; and its simple projective characterization was later given by C. Segre [4].

consider the asymptotic curves Γ, Γ_1 of the surface S_1 at the point O. The expansions of these asymptotic curves at the point O are found, after a simple calculation, to be

$$\Gamma: \quad y = -2px^3 + \cdots, \qquad z = -(3p/l)x^2 + \cdots; \tag{8}$$

$$\Gamma_1: \quad y = -2qz^3 + \cdots, \qquad x = -(3q/l)z^2 + \cdots. \tag{9}$$

Let K be any five-point quadric cone of the asymptotic curve Γ at the point O with O for vertex and with an asymptotic tangent of the surface S_2 at the point O for a generator, then making use of equations (8) we can easily obtain its equation, namely,

$$z^2 + (9p/2l^2)xy \pm (9p/2l^2)(-n/m)^{1/2}y^2 + kxz = 0, \tag{10}$$

where k is arbitrary. Similarly, we have a five-point quadric cone K_1 of the asymptotic curve Γ_1 at the point O such that it has O for vertex and the line t_2 for a generator and is tangent to the plane τ_2, $z=0$, along t_2. The equation of this cone K_1 is

$$x^2 + (9q/2l^2)yz = 0. \tag{11}$$

From equations (10), (11) it follows that the cone projecting the curve of intersection of the cones K, K_1 from the line t_2 consists of the four planes:

$$9q^2z^4 - 18kl^2qx^2z^2 - 9pqx^3z \pm 2l^2p(-n/m)^{1/2}x^4 = 0, \tag{12}$$

which are intersected by the plane τ_1 in four lines λ_i $(i=1, \cdots, 4)$ with equations

$$z + \alpha_i x = 0, \qquad y = 0 \qquad (i = 1, \cdots, 4), \tag{13}$$

where the coefficients α_i satisfy the relation

$$\alpha_1\alpha_2\alpha_3\alpha_4 = \pm (2l^2p/9q^2)(-n/m)^{1/2}. \tag{14}$$

On the other hand, associated with the point O of the surface S_1 there are three tangents of Darboux, whose equations may be found by making use of equations (8), (9) and Bompiani's result [1] that they are the three principal lines of the asymptotic curves Γ, Γ_1 at the point O. The result is

$$px^3 + qz^3 = 0, \qquad y = 0. \tag{15}$$

Let d be any one of the tangents (15) of Darboux, and let D_i denote the cross ratio of the four lines t, t_1, d, λ_i $(i=1, \cdots, 4)$; then

$$D_i = (tt_1, d\lambda_i) = (1/\alpha_i)(p/q)^{1/3} \qquad (i = 1, \cdots, 4). \tag{16}$$

A reference to equations (5), (14), (16) suffices to substantiate that *the invariant J can be expressed in terms of the four cross ratios D_1, D_2, D_3, D_4 as follows*:

$$J = -(2/9)^6(D_1D_2D_3D_4)^6. \tag{17}$$

II. Two surfaces with distinct tangent planes and a common asymptotic tangent at an ordinary point

4. Derivation of an invariant. Finally, suppose that S_1, S_2 be two surfaces in ordinary space intersecting at an ordinary point O with distinct tangent planes τ_1, τ_2, whose line of intersection t stands for one asymptotic tangent of both surfaces S_1, S_2 at the point O; and that t_1, t_2 be respectively the other asymptotic tangents of the surfaces S_1, S_2 at the point O. If we choose the point O to be the origin, the lines t, t_2, t_1 to be respectively the axes x, y, z of a general non-homogeneous projective coordinate system, then the power series expansions of the surfaces S_1, S_2 in the neighborhood of the point O may be written in the form (1) and

$$S_2: \quad z = mxy + \cdots. \tag{18}$$

In a way similar to the foregoing we can easily show that *the quantity*

$$I = p^4q^2/l^3m^3 \tag{19}$$

is a projective invariant associated with the surfaces S_1, S_2 at the point O.

5. A projective characterization of the invariant I. Let Γ, Γ_1 be the asymptotic curves of the surface S_1 at the point O whose tangents are t, t_1 respectively. Among the four-point quadrics of the asymptotic curve Γ at the point O we can determine a two-parameter family such that every one of the family has t_2 for a generator and has at the point O contact of the second order with the surface S_2. By means of equations (8), (18) it is easy to obtain the equation of a general quadric of this family, namely,

$$z + (3p/l)x^2 - mxy + Eyz + Fz^2 = 0, \tag{20}$$

where E, F are arbitrary. The quadric (20) is cut by the plane τ_2 in the line t_2 and a line with equations

$$3px - lmy = 0, \qquad z = 0. \tag{21}$$

If any five-point quadric cone, with the point O for vertex, of the

asymptotic curve Γ at O passes through the line (21), then it must have the equation:

$$z^2 + (9p/2l^2)xy - (3m/2l)y^2 + kyz = 0, \tag{22}$$

where k is arbitrary. The cone projecting the curve of intersection of the cones (11), (22) from the line t_2 consists of the four planes:

$$27q^2z^4 - 6kl^2qx^2z^2 - 27pqx^3z - 2l^3mx^4 = 0, \tag{23}$$

which are intersected by the plane τ_1 in four lines μ_i $(i=1, \cdots, 4)$:

$$z + \beta_i x = 0, \quad y = 0 \qquad (i = 1, \cdots, 4), \tag{24}$$

where the coefficients β_i satisfy the relation

$$\beta_1\beta_2\beta_3\beta_4 = -2l^3m/27q^2. \tag{25}$$

If d be any one of the three tangents of Darboux associated with the point O of the surface S_1 and if D_i denotes the cross ratio of the four lines t, t_1, d, μ_i $(i=1, \cdots, 4)$, then from equations (15), (24) it follows that

$$D_i = (tt_1, d\mu_i) = (1/\beta_i)(p/q)^{1/3} \qquad (i = 1, \cdots, 4). \tag{26}$$

A reference to equations (19), (25), (26) suffices to show that *the invariant I can be expressed in terms of the four cross ratios D_1, D_2, D_3, D_4 as follows*:

$$I = -(2/27)^3(D_1D_2D_3D_4)^3. \tag{27}$$

References

1. E. Bompiani, *Invarianti d'intersezione di due curve sghembe*, Rendiconti dei Lincei (6) vol. 14 (1931) pp. 456–461.

2. C. C. Hsiung, *An invariant of intersection of two surfaces*, to be published in the American Journal of Mathematics.

3. R. Mehmke, *Einige Sätze über die räumliche Collineation und Affinität, welche sich auf die Krümmung von Curven und Flächen beziehen*, Schlömilchs Zeitschrift für Mathematik und Physik vol. 36 (1891) pp. 56–60; *Über zwei die Krümmung von Curven und das Gauss'sche Krümmungsmass von Flächen betreffende charakteristische Eigenschaften der linearen Punkttransformationen*, ibid. vol. 36 (1891) pp. 206–213.

4. C. Segre, *Su alcuni punti singolari delle curve algebriche, e sulla linea parabolica di una superficie*, Rendiconti dei Lincei (5) vol. 6 (1897) pp. 168–175.

5. H. J. S. Smith, *On the focal properties of homographic figures*, Proc. London Math. Soc. vol. 2 (1869) pp. 196–248.

6. B. Su, *A new invariant of intersection of two hypersurfaces*, to be published in the Revista, Universidad Nacional de Tucumán (A).

National University of Chekiang

SOME REMARKS ON CONNECTED SETS

P. ERDÖS

This note will consist of a few disconnected remarks on connected sets.

1. Swingle[1] raised the question whether the plane is the sum of c disjoint biconnected sets. The answer as we shall show is affirmative.

First we construct a biconnected set A with a dispersion point x such that any two points of $A-x$ can be separated. (The first such set was constructed by Wilder.[2] Our construction will be very similar to that of Burton Jones.[3])

Our biconnected set will contain the origin $\mathfrak{O}$, at most one other point on lines through the origin with irrational slopes, and no point other than the origin on lines with rational slopes. Further it will contain at least one point on every cut of the plane. It is easy to see that such a set exists. Every cut of the plane contains a closed subset which also cuts the plane, and the power of closed sets is known to be c. Let us well order the closed cuts $C_1, C_2, \cdots, C_\gamma, \cdots, \gamma<\Omega_\alpha$, where Ω_α is the least ordinal number of power c. We construct A as follows: $\mathfrak{O}$ belongs to A. We shall choose a point x_γ on C_γ and we shall have $A=\mathsf{U}x_\gamma$. We shall determine x_γ by transfinite induction. Suppose we have already determined x_δ, $\delta<\gamma$, we determine x_γ as follows: if C_γ contains $\mathfrak{O}$ then $x_\gamma=\mathfrak{O}$. If C_γ does not contain $\mathfrak{O}$ then clearly C_γ has to intersect c lines through $\mathfrak{O}$. Therefore we can find a point $x_\gamma\in C_\gamma$ such that $(\mathfrak{O}, x_\gamma)$ has irrational slope and does not go through any other point x_δ, $\delta<\gamma$. (We denote by (a, b) the line through the points a and b.) This way we construct A. Clearly A is biconnected. First of all A is connected since it intersects every cut of the plane. Also any two points of $A-\mathfrak{O}$ can be separated since the two points x_1 and x_2, say, are on different irrational lines through the origin, and a rational line, which of course does not intersect $A-\mathfrak{O}$, will separate them. This completes the proof. The origin we call the center of A.

Now we shall split the plane into the sum of c such disjoint sets A_γ, $\gamma<\Omega_\alpha$, Ω_α the smallest ordinal of power c. Let us well order the points x_γ of the plane and the closed cuts C_γ of the plane. Our first step is to select x_1 as the center of A_1 and a suitable point of A_1 on C_1.

Received by the editors February 2, 1944.

[1] P. M. Swingle, Amer. J. Math. vol. 54 (1932) p. 532.

[2] R. L. Wilder, Bull. Amer. Math. Soc. vol. 33 (1927) p. 423.

[3] Burton Jones, Bull. Amer. Math. Soc. vol. 48 (1942) p. 115.

(By suitable we mean that it does not lie on any line with rational slope through x_1.) Our second step is to select the first point which has not yet been selected as center of A_2, then select suitable points on C_1 and C_2 for A_2 and on C_2 for A_1. Suppose that this construction has been already carried out for all $\delta<\gamma$. Then the γth step consists of the following construction: First we take the first x_ε which has not yet been chosen as center of A_γ. Then we choose suitable points on all the C_δ, $\delta\leqq\gamma$, for A_γ and suitable points on C_γ for all the A_δ, $\delta<\gamma$. It is easy to see that this construction can be carried out since before the γth step we have chosen less than c points x_j, and a cut C_γ if it does not go through the origin intersects c lines through the origin. (If the cut C goes through the origin of one or more of the A_γ's then of course we can choose the origin.) Thus the construction can be carried out for all ordinals $\gamma<\Omega_a$. Hence we get c sets A_γ where $\bigcup A_\gamma$ is the whole plane, $A_{\gamma_1}\bigcap A_{\gamma_2}=0$ and the A_γ are all biconnected, which completes the proof of the theorem.

A simple modification of this proof would give the following result: Let $m\leqq c$ be any cardinal number greater than 2, then the plane can be expressed as the sum of m disjoint biconnected sets.[4]

ii. Let C be any connected set. Knaster and Kuratowski[5] proved that there exists a proper connected subset of C. (Single points, of course, do not count as connected sets.) Their proof is very simple. Let $p\in C$ be arbitrary. If $C-p$ is connected our result is proved. If $C-p$ is not connected it can be written in the form $U+V$ where U and V are separated. But then it is easy to see that both $U+p$ and $V+p$ are connected.

It seems likely that the following result is true: every connected set C contains a connected subset C' such that $C-C'$ has power c. I am unable to prove this. It is easy to see that if a set C does not satisfy this conjecture it has to be both biconnected and widely connected, thus must be rather pathological.

A. Stone[6] proved that every connected set C has a connected subset C' such that $C-C'$ is infinite. Proof: First we can assume that $C-p$ is connected for every $p\in C$, for if not then at least one of the sets $U+p$ or $V+p$ has a complement of power c. (U and V were defined in the previous paragraph.) Similarly for every sequence $p_1, p_2, \cdots, p_n$, $C-p_1-p_2-\cdots-p_n$ is connected. Choose $p_1, p_2, \cdots$

[4] Swingle proved (ibid.) that the Euclidean n space is the sum of r biconnected sets if r is any finite number not less than $n+1$, also that it is the sum of $\aleph_0$ biconnected sets. He also showed that n-space is not the sum of n or less biconnected sets.

[5] Knaster-Kuratovski, Fund. Math. vol. 2 (1921) p. 206.

[6] Arthur Stone, oral communication.

such that it will converge to a point $p \in C$. Then $C - \sum_{i=1}^{\infty} p_i$ is connected. For if $\sum_{i=1}^{\infty} p_i$ would separate C, a closed subset of it would also separate C. But a closed subset of $\sum_{i=1}^{\infty} p_i$ is finite, and so by assumption can not separate C, and this completes the proof.

In this connection it might be of some interest to construct a connected set A with the following properties: (1) no finite subset disconnects A, (2) every countable dense subset totally disconnects A. For our construction we shall have to use the continuum hypothesis.

First we need the following lemma: Let S_1 and S_2 be two countable disjoint sets. S_1 is everywhere dense in the plane. Then there exist countably many closed Jordan curves J_r such that $J_r \cap S_1$ is dense on J_r, $J_r \cap S_2 = 0$, and to any two points p and q there exists an r such that J_r separates p and q. The proof does not present any difficulties and can be omitted.

And now to our construction.[7] First we well order the countable dense subsets of the plane, $D_1, D_2, \cdots, D_\alpha, \cdots$ (their number is clearly $c = \aleph_1$; the continuum hypothesis has been assumed). Also we well order all the closed connected cuts of the plane C_δ, again clearly $\delta < \Omega_1$ (it is well known that every cut of the plane contains a closed connected cut). We shall now construct our set A by transfinite induction. Let Δ be any given countable dense set, put $\Delta \subset A$. Our first step is to define a countable subset P_1 of C_1. If $\Delta \cap C_1$ is infinite we put $P_1 = 0$, if $C_1 \cap \Delta$ is finite, we choose an arbitrary countably infinite $P_1 \subset C_1$ and put $P_1 \subset A$. Consider next the smallest α such that $D_\alpha \subset \Delta + P_1$. Put $S_1 = D_\alpha$ and $S_2 = \Delta + P_1 - D_\alpha$. Then by our lemma there exists a countable collection of closed Jordan curves $J_r^{(1)}$ such that $J_r^{(1)} \cap D_\alpha$ is dense on $J_r^{(1)}$ and $J_r^{(1)} \cap \Delta + P_1 - D_\alpha = 0$ and any two points p and q are separated by some $J_r^{(1)}$. Now we make the condition that no point of any P_β $(\beta > 1)$ should lie on any J_r^{β}. Suppose we carried out our construction for any $\gamma < \delta$, we shall show that we can carry it out for $\gamma = \delta$. Consider C_δ; if $C_\delta \cap \Delta + \sum_{\beta<\delta} P_\beta$ is infinite, then $P_\delta = 0$. If $C_\delta \cap \Delta + \sum_{\beta<\delta} P_\beta$ is finite, then we choose a countably infinite subset P_δ of C_δ, such that P_δ does not meet $\sum_{\lambda<\delta} \sum_{r=1}^{\infty} J_r^{\lambda}$. In other words P_δ does not meet any of countably many Jordan curves (we have assumed the continuum hypothesis). We have to prove that such a choice of P_δ is possible. By assumption $(\Delta + \sum_{\beta<\delta} P_\beta) \cap C_\delta$ is finite. For each of the Jordan curves J_r^{λ} (with $\lambda < \delta$) $J_r^{\lambda} \cap \Delta + \sum_{\beta<\delta} P_\beta$ is dense on J_r^{λ}, therefore $C_\delta \cap J_r^{\lambda}$ is nowhere dense on C_δ (for if not, since C_δ is closed, it would contain a portion of J_r^{λ} and therefore $C_\delta \cap (\Delta + \sum_{\beta<\delta} P_\beta)$ would be infinite, as $\Delta + \sum_{\beta<\delta} P_\beta$ is dense on J_r^{λ}).

[7] Our construction will be similar to Miller's construction of a biconnected set without a dispersion point. (Fund. Math. vol. 29 (1937) p. 123.)

Therefore $C_\delta \cap \sum_{\lambda<\delta} \sum_{r=1}^{\infty} J_r^\lambda$ is of first category on C_δ, and thus there are c points of C not belonging to $\sum_{\lambda<\delta} \sum_{r=1}^{\infty} J_r^\lambda$; therefore we can choose an infinite $P_\delta \subset C_\delta$, $P_\delta \cap \sum_{\lambda<\delta} \sum_{r=1}^{\infty} J_r^\lambda = 0$ without difficulty. Next consider the smallest α such that $D_\alpha \subset \Delta + \sum_{\beta \leq \delta} P_\beta$ and D_α has not yet been dealt with in any of the previous steps. We then construct by our lemma countably many closed Jordan curves J_r^δ such that $D_\alpha \cap J_r^\delta$ is dense on J_r^δ, $J_r^\delta \cap (\Delta + \sum_{\beta<\delta} P_\beta - D_\alpha) = 0$, and any two points p and q can be separated by some J_r^δ. This completes the δth step. Thus our construction can be carried out through all ordinals of the second number class and will exhaust all the C_α's and D_α's. Consider now the set $A = \Delta + \sum_\beta P_\beta$. We claim that no finite set disconnects it and that any countable dense set totally disconnects it. First of all every cut of the plane intersects our set in an infinite set, thus no finite set can disconnect it. On the other hand let D_α be some countable dense set of A (clearly A is dense in the plane). At some step in our construction we had to deal with D_α and it is clear that $\Delta + \sum_\beta P_\beta - D_\alpha$ is totally disconnected, since if p and q are any two points there exists a J_r separating them and $(\Delta + \sum_{\beta<\alpha} P_\beta - D_\alpha) \cap J_r = 0$. q.e.d.

By a slight modification of Miller's[8] construction of a biconnected set without a dispersion point, we can construct a connected set A dense in an indecomposable continuum, such that there exists an open set O of the plane with $O \cap A$ having power c and if B is any connected subset of A then $A - B$ is nowhere dense in $O \cap A$. Clearly such a set is biconnected and in fact it can be made to have no dispersion point. By a further slight modification of Miller's construction we can construct a biconnected set without a dispersion point such that if $A_1, A_2, \cdots$ is any countable collection of connected subsets of A, then $\bigcup_i (A - A_i) \neq A$ or $\bigcap_i A_i \neq 0$.

I can not decide the question whether there exists a connected set such that the complement of every connected subset of it is nowhere dense in it.

The following problems may be of some interest: Is it true that every connected set contains a connected subset not homeomorphic to it? (Points do not count as connected sets.) Every known connected set (which is subset of a Euclidean space) contains at least three different types of connected subsets. The number three can not be improved as is shown by the arc.

If a topological space does not satisfy any separation axioms the following example communicated to me by S. Eilenberg shows that the above conjecture is not true: Let the space be the integers, the

[8] Miller, ibid.

closed sets are the finite sets. Every infinite set is connected and they are clearly all homeomorphic.

Is it true that every connected set of dimension n contains a connected subset of dimension $n-1$?

Is it true that every connected set of dimension greater than 1 contains 2^c connected subsets?

Perhaps the following theorem which A. Stone and I proved might be of some interest: Let A be a totally disconnected set of power m (in a separable space), then if m is not the sum of countably many smaller cardinals, A can be written as the sum of two separated sets of power m. If m is the sum of $\aleph_0$ smaller cardinals the theorem is in general false.

PROOF. Suppose first $m \neq \sum_{k=1}^{\infty} m_k$, $m_k < m$. Denote by A' the set of those points of A for which every neighborhood contains m points of A. Now $A-A'$ has power less than m, since by Lindelöf's theorem $A-A'$ can be covered by countably many open sets each of which contains less than m points, thus $A-A'$ also contains less than m points. Since A is totally disconnected $A' = U+V$, where U and V are open-closed sets. Thus both of them have power m. Let $p \in A-A'$, the distance of p from U be $f(p)$, from V, $\phi(p)$, $p \in U_\alpha$ if $f(p) \leqq \alpha\phi(p)$, $p \in V_\alpha$ if $f(p) \geqq \alpha\phi(p)$. Since the power of $A-A'$ is less than $m \leqq c$ there exists an α such that $U_\alpha \cap V_\alpha = 0$. But then $U+U_\alpha$ and $V+V_\alpha$ are two separated sets of power m whose sum is A; this completes the proof.

If $m = \sum_{k=1}^{\infty} m_k$, $m_k < m$, we define A as follows: A contains m_k points in $(1/k, 1/k+i)$, 0 belongs to A. Clearly A is totally disconnected (its power is less than c, since c is not the sum of m smaller cardinals). Obviously A is not the sum of two separated sets of power m.

PURDUE UNIVERSITY